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இலங்கைப் பரீட்சைத் திணைக்களம் இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2025
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General Certificate of Education (Adv. Level) Examination, 2025

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

Part B

* Answer **five** questions only.

11. (*a*) Let $f(x) = x^2 - 2kx - k^2 - 1$, for $x \in \mathbb{R}$, where $k \in \mathbb{R}$.

Show that the equation $f(x) = 0$ has two non-zero roots that are real and distinct.

Let α and β be the roots of $f(x) = 0$, and let $r = \dfrac{1}{2\alpha}$ and $s = \dfrac{1}{2\beta}$.

Show that the quadratic equation with r and s as its roots is $4(k^2+1)x^2 + 4kx - 1 = 0$ and that $|r-s| = \dfrac{\sqrt{2k^2+1}}{k^2+1}$.

Deduce that the horizontal distance between the two points of intersection of the graphs of $y = x^3 + 9x^2 + 3x + 1$ and $y = x^3 + x^2 - x + 2$ is $\dfrac{\sqrt{3}}{2}$.

(*b*) (i) Let $a \in \mathbb{R}$. Show that if $(x-a)$ is a factor of a polynomial $p(x)$, then there exists a polynomial $s(x)$ such that $p(x) - (x-a)p'(x) = (x-a)^2 s(x)$, where $p'(x)$ is the derivative of $p(x)$.

(ii) Let $g(x) = x^3 - \lambda x^2 - 2x - (x-2)(3x^2 + \mu x - 2)$ for $x \in \mathbb{R}$, where $\lambda, \mu \in \mathbb{R}$. It is given that $(x-2)$ is a factor of $g(x)$ and that the remainder when $g(x)$ is divided by $(x-1)$ is -3. Show that $\lambda = 1$ and $\mu = -2$.
Using (i) above, **deduce** that $(x-2)^2$ is a factor of $g(x)$.

12. (*a*) Five books that are different from each other, are to be distributed among three students *A*, *B* and *C*.

Find the number of different ways in which

(i) *A* receives exactly 2 books, *B* receives exactly 2 books and *C* receives exactly 1 book,

(ii) each student receives at least 1 book.

[see page eight

(*b*) Let $U_r = \dfrac{7r+4}{r(r+1)(r+2)}$, $f(r) = \dfrac{A}{r}$ and $g(r) = \dfrac{B}{r+1}$ for $r \in \mathbb{Z}^+$, where A and B are real constants.

Find the values of A and B such that $U_r = [f(r) - f(r+2)] + [g(r) - g(r+1)]$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum_{r=1}^{n} U_r = \dfrac{9}{2} - \dfrac{2}{n+1} - \dfrac{5}{n+2}$ for $n \in \mathbb{Z}^+$.

Show further that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Find the value of the real constant m such that $\lim_{n\to\infty} \left(\sum_{r=1}^{2n} U_r + m \sum_{r=1}^{n-1} U_{n-r} \right) = 18$.

13. (*a*) Let $\mathbf{A} = \begin{pmatrix} 2 & a \\ 3 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & a \\ 1 & 4 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4+a & 7 \\ 6 & 7 \\ 3a & 4 \end{pmatrix}$, where $a \in \mathbb{R}$.

Find the values of a such that $\mathbf{A}^{-1}$ exists.

Find $\mathbf{AB}$ in terms of a.

Determine the value of a such that $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{C}$.

For this value of a, show that $\mathbf{A} - \mathbf{A}^{-1} = 3\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Let $w = -\dfrac{\sqrt{2}(3+i)}{(1+2i)}$. Show that $|w| = 2$ and Arg $w = \dfrac{3\pi}{4}$.

Let $z \in \mathbb{C}$ be such that $|z| = 2$ and Arg $z = \dfrac{\pi}{3}$.

Also, let A and B be the points representing the complex numbers z and w respectively, on an Argand diagram. Show that $AB^2 = 8 + 2\sqrt{2} - 2\sqrt{6}$.

Let C be the mid-point of AB. Using the triangle AOC, where O is the origin, **deduce** that $\sin^2\left(\dfrac{5\pi}{24}\right) = \dfrac{1}{8}\left(4 + \sqrt{2} - \sqrt{6}\right)$.

(*c*) Let $m \in \mathbb{Z}^+$. It is given that $(1 + \sqrt{3}\,i)^{3m}(1 + i)^8 = 2^{3m+4}$.

Find the least value of m.

[*see page nine*

14. (*a*) Let $f(x)=\dfrac{x^2+x+p}{(x-1)^2}$ for $x \in \mathbb{R} - \{1\}$, where $p \in \mathbb{R}$. It is given that the graph of $y=f(x)$ intersects its horizontal asymptote at a point with the x-coordinate $-\dfrac{1}{3}$. Show that $p=2$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = -\dfrac{(3x+5)}{(x-1)^3}$ for $x \in \mathbb{R} - \{1\}$.

Hence, find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

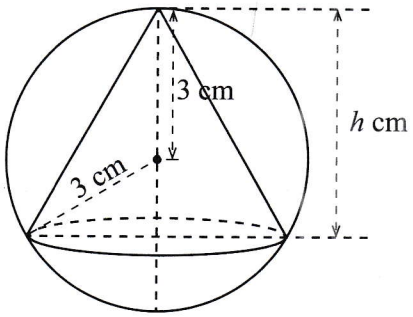
Also, find the coordinates of the turning point of the graph of $y=f(x)$.

It is given that $f''(x)$, the second derivative of $f(x)$ is given by $f''(x) = \dfrac{6(x+3)}{(x-1)^4}$ for $\mathbb{R} - \{1\}$.

Show that the graph of $y=f(x)$ has a point of inflection at $(-3, \frac{1}{2})$.

Sketch the graph of $y=f(x)$ indicating the turning point, the asymptotes and the point of inflection.

(*b*) A right circular cone is to be inscribed in a sphere of radius 3 cm, as shown in the figure.



Let h cm be the height of the cone. Show that the volume of the cone V cm^3 is given by $V=\dfrac{\pi}{3}\left(6h^2-h^3\right)$.

Also, show that the largest such cone that can be inscribed in the sphere is obtained when $h = 4$.

15. (*a*) Let $k \in \mathbb{R}$. Find $\displaystyle\int \frac{\sqrt{x}}{(1-k^2x)}\,dx$.

(*b*) Show that $\dfrac{d}{dx}\left\{\ln\left(\dfrac{1+\sin 2x}{\cos 2x}\right)\right\} = \dfrac{2}{\cos 2x}$ for $0 < x < \dfrac{\pi}{4}$.

Using integration by parts, find $\displaystyle\int (\cos 2x)\ln\left(\frac{1+\sin 2x}{\cos 2x}\right)dx$.

(*c*) Using the formula $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where a is a constant, show that

$$\int_0^{\frac{\pi}{6}} \frac{\cos(x+\frac{\pi}{3})}{\sin x+\cos(x+\frac{\pi}{3})}\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin x}{\sin x+\cos(x+\frac{\pi}{3})}\,dx.$$

Deduce that $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\cos(x+\frac{\pi}{3})}{\sin x+\cos(x+\frac{\pi}{3})}\,dx = \frac{\pi}{12}$.

[*see page ten*

16. Let O be the origin, $A \equiv (1, 2)$ and $B \equiv \left(\frac{5}{2}, \frac{5}{4}\right)$. Find the equations of the angle bisectors of the angles $A\hat{O}B$ and $O\hat{A}B$, and show that these angle bisectors intersect at the point $D \equiv \left(\frac{5}{4}, \frac{5}{4}\right)$.

Find the perpendicular distance from D to the line OA.

Write down the equation of the circle C_1 that touches all three sides of the triangle OAB.

Suppose that the circle C_1 touches OA and AB at the points E and F, respectively. Show that the equation of the circle C_2 that goes through the points A, E and F is given by $4x^2 + 4y^2 - 9x - 13y + 15 = 0$.

Determine whether the intersection of the circles C_1 and C_2 is orthogonal.

17. (*a*) Write down $\sin(A + B)$ in terms of $\sin A$, $\sin B$, $\cos A$ and $\cos B$.

Deduce that $\sin 2\theta = 2\sin\theta\cos\theta$.

Let $0 < \theta < \frac{\pi}{2}$. Express the equation $\cot\theta - 2\tan\theta = \sin 2\theta$ in the form $a\cos^4\theta + b\cos^2\theta + c = 0$, where a, b and c are real constants.

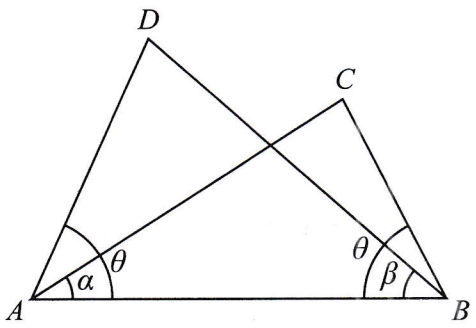
Hence, show that $\theta = \cos^{-1}\sqrt{\frac{\sqrt{17}-1}{4}}$.

(*b*) Four distinct points A, B, C and D on a plane are such that $B\hat{A}D = A\hat{B}C = \theta$ and $3AD = 4BC$.

Let $B\hat{A}C = \alpha$ and $A\hat{B}D = \beta$. (See the figure.)

Using the Sine Rule, show that $\frac{BC}{AD} = \frac{\sin\alpha\sin(\theta+\beta)}{\sin\beta\sin(\theta+\alpha)}$.

Deduce that $\cot\theta = 3\cot\alpha - 4\cot\beta$.



(*c*) Solve the following simultaneous equations for x and y:

$\sin^{-1}\sqrt{x} = \cos^{-1}\sqrt{y}$

$\tan(\tan^{-1}3x - \tan^{-1}2y) + \tan(\tan^{-1}3y - \tan^{-1}2x) = 1$.

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