Part A

1. Let $u_1 = 2$ and $u_{n+1} = u_n + 2n$ for all $n \in \mathbf{Z}^+$. Using the **Principle of Mathematical Induction**, prove that $u_n = n^2 - n + 2$ for all $n \in \mathbf{Z}^+$.

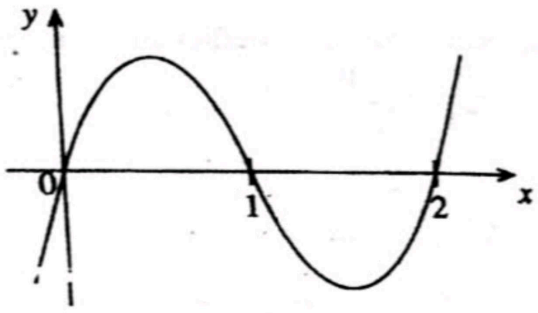
2. Sketch the graphs of $y = \frac{1}{2}|x+1|$ and $y = |2x| - 1$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $4|x| - |x+1| \leq 2$.

3. Shade in an Argand diagram, the region S consisting of the points that represent the complex numbers z satisfying the inequalities $|z-3| \leq 3$ and $0 \leq \text{Arg}\, z \leq \frac{\pi}{6}$.
Also, find the area of S.

4. Show that the coefficient of x^{-3} is three times the coefficient of x^{-2} in the Binomial expansion of $\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{x}}\right)^{10}$.

5. Show that $\lim_{x \to 1} \frac{(1+\cos \pi x)}{(x-1)^2} = \frac{\pi^2}{2}$.

6. Let $f(x) = x(x-1)(x-2)$. A sketch of the graph of $y = f(x)$ is given in the adjoining figure:



Find $\int_0^2 |f(x)|\,dx$.

7. Let $a>0$ and let C be the curve given parametrically by $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$ for $0<\theta<\pi$. Show that $\frac{dy}{dx}=\cot\frac{\theta}{2}$.

Find $\frac{d^2y}{dx^2}$ in terms of θ and **deduce** that C is concave downwards.

8. Let l_1 and l_2 be the straight lines given by $2y-x-2=0$ and $y+2x-6=0$, respectively. Find the equation of the straight line l_3 that passes through the point of intersection of l_1 and l_2, and the point $(-4,4)$.

Also, show that l_3 is a bisector of an angle between l_1 and l_2.

9. Find the equations of the circles that touch the circle $x^2 + y^2 = 4$ externally and also touch the lines $y = 0$ and $y = 2$.

10. Solve for x: $\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x + \frac{\pi}{4}\right) = \cos\frac{\pi}{24}$.

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අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2023 (2024)
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Combined Mathematics	**I**

10 E I

Part B

* Answer **five** questions only.

11. (*a*) Let $f(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$, where $a, b, c \in \mathbb{R}$ with $a > 0$.

Show that the minimum value of $f(x)$ is $-\frac{\Delta}{4a}$, where $\Delta = b^2 - 4ac$.

Let p and q be positive real numbers and $r \in \mathbb{R}$. Also, let $g(x) = px^2 + 2\sqrt{pq}\,x + qr$ for $x \in \mathbb{R}$.

It is given that the equation $g(x) = 0$ has no real roots. Show that $r > 1$.

Now, the minimum value of $g(x)$ is given to be q. Show that $r = 2$.

If the straight line $y = x + 1$ is the tangent line to the curve $y = g(x)$ with $r = 2$ at the point $(0, 1)$, find the values of p and q.

(*b*) Let $a \in \mathbb{R}$ and $p(x)$ be a polynomial of degree 4. Show that if $(x - a)$ is a factor of both $p(x)$ and $p'(x)$, then $(x - a)^2$ is a factor of $p(x)$, where $p'(x)$ is the derivative of $p(x)$ with respect to x.

Let $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$ for $x \in \mathbb{R}$. **Deduce** that $(x - 2)^2$ is a factor of $f(x)$.

Find the value of $f(-1)$ and factorize $f(x)$ completely.

12. (*a*) A committee consisting of 4 men and 4 women is to be selected from a group of 8 men and 6 women.

(i) Find the number of different ways in which the committee can be selected.

(ii) Suppose that such a committee was selected. Find the number of different ways in which those committee members can be seated in a row if no two women can be seated next to each other.

(*b*) It is given that $\sum_{r=1}^{n} U_r = \frac{n}{4}(n+1)(n+2)(n+3)$ for all $n \in \mathbb{Z}^+$.

Show that $U_n = n(n+1)(n+2)$ for all $n \in \mathbb{Z}^+$.

Let $V_r = \frac{1}{U_r}$ for $r \in \mathbb{Z}^+$.

Find the real constants A and B such that $V_r = \frac{A}{r(r+1)} + \frac{B}{(r+1)(r+2)}$ for all $r \in \mathbb{Z}^+$.

Hence or otherwise, show that $\sum_{r=1}^{n} V_r = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$ for $n \in \mathbb{Z}^+$.

Show further that the infinite series $\sum_{r=1}^{\infty} V_r$ is convergent and find its sum.

Find $m \in \mathbb{Z}^+$ such that $\sum_{r=m}^{\infty} V_r = \frac{1}{24}$.

13. (*a*) Let $a \in \mathbb{R}$ and $\mathbf{A} = \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}$. Show that $\mathbf{A}^{-1}$ exists, and write down $\mathbf{A}^{-1}$.

Let $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

(i) Find the value of a such that $\mathbf{A}^{-1}\mathbf{B}^{\mathbf{T}} = -\frac{1}{5}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$.

(ii) Find the matrix $\mathbf{C}$ such that $\mathbf{BC} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 5 \end{pmatrix}$.

(*b*) Let $z \in \mathbb{C}$. Define $\overline{z}$, the complex conjugate of z, and $|z|$, the modulus of z.

Show that if $|z| = 1$, then $\overline{z} = \frac{1}{z}$ and **deduce** that $|z - w| = |1 - \overline{z}w|$ for any $w \in \mathbb{C}$.

Now, let $z = \frac{1}{2}(1 + \sqrt{3}i)$. Find $|z|$ and Arg z.

Let $w \in \mathbb{C}$ be such that $|w| < 1$ and Arg $w = \alpha$, where $0 < \alpha < \frac{\pi}{3}$.

By choosing one such complex number w, plot the points that represent 1, z, w and $\overline{z}w$ in an Argand diagram and explain geometrically why $|z - w| = |1 - \overline{z}w|$.

(*c*) Let $n \in \mathbb{Z}^+$. Find the smallest value of n such that the real part of $\frac{\left(\cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}\right)^n}{\left(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}\right)^7}$ is $\frac{1}{2}$.

14.(a) Let $a, p, q \in \mathbb{R}$ and $p < q$.

Let $f(x) = \dfrac{(ax+1)(x+2)}{(x-p)(x-q)}$ for $x \in \mathbb{R} - \{p, q\}$.

It is given that the vertical asymptotes of the graph of $y = f(x)$ are $x = 1$ and $x = -4$. Write down the values of p and q. Given that $y = 1$ is a horizontal asymptote of the graph of $y = f(x)$, show that $a = 1$.

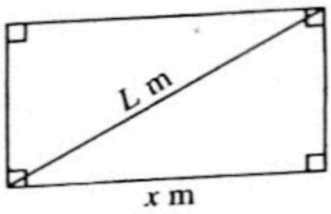
For these values of a, p and q, find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Let $g(x) = f(x) + 1$.

Sketch the graph of $y = g(x)$, indicating asymptotes and turning points.

State the range of $g(x)$.

(b) It is required to build a fence along a diagonal of a rectangular region of area k m^2. Let x m be the length of the rectangle (See the figure). Show that the length L m of the fence is given by $L^2 = x^2 + \dfrac{k^2}{x^2}$.



Hence, show that L is minimum when $x = \sqrt{k}$.

15.(a) Let $k \in \mathbb{R}$. Find $\displaystyle\int \frac{1}{x^2(x-k)}\,dx$.

(b) By applying integration by parts to $\displaystyle\int_1^{e^{\frac{\pi}{2}}} x\sin(\ln x)\,dx$ or otherwise show that

$$\int_1^{e^{\frac{\pi}{2}}} x\{2\sin(\ln x) + \cos(\ln x)\}\,dx = e^{\pi}.$$

(c) Let $k > 0$. Show that $\dfrac{d}{dx}\left\{\left(k\sqrt{x} - 1\right)e^{k\sqrt{x}}\right\} = \dfrac{k^2}{2}e^{k\sqrt{x}}$ for $x > 0$.

Also, let $I_k = \displaystyle\int_1^4 e^{k\sqrt{x}}\,dx$. Show that $I_k = \dfrac{2}{k^2}\left\{(2k-1)e^{2k} - (k-1)e^{k}\right\}$.

Let S be the region enclosed by the curves $y = e^{\sqrt{x}}$, $x = 1$, $x = 4$ and $y = 0$.

Show that the area of S is $2e^2$.

Also, find the volume of the solid generated by rotating the region S about the x-axis through 2π radians.

16. Let $m \in \mathbb{R}$ and let l be the line through the point $A \equiv (3, 1)$ with gradient m.

Write down the equation of l in terms of m.

Show that there exists two tangents to the circle $S_1 \equiv 5x^2 + 5y^2 - 10x + 10y + 6 = 0$ through the point A, and find the acute angle between them.

Let B and D be the points at which these tangents touch the circle $S_1 = 0$, and C be the centre of $S_1 = 0$.

Show that $ABCD$ is a cyclic quadrilateral and find the equation of the circle through the points A, B, C and D.

Find the equation of the chord of contact BD and find the equation of the circle passing through B and D that intersects the circle $S_1 = 0$ orthogonally.

17. (*a*) Show that $\cos^2\theta + \sin^2\theta = 1$ for $\theta \in \mathbb{R}$.

Find all values of x in the interval $[0, 2\pi)$ that satisfy the equation $\cos^2 x - 1 = \sin^2 x + 3\cos x$.

(*b*) Let ABC be a triangle. In the usual notation, using the result that $A + B + C = \pi$, show that $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ and $\cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}$.

Deduce that $\tan\frac{B}{2} + \tan\frac{C}{2} = \cos\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}$ and $1 - \tan\frac{B}{2}\tan\frac{C}{2} = \sin\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}$.

Hence, show that $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$.

(*c*) Solve $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{3\pi}{4}$ for $x \in \mathbb{R}$.

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