අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2023 (2024) අගෝස්තු
General Certificate of Education (Adv.Level) Examination, August 2023 (2024)
Combined Maths

I Paper

Part A

01. For n = 1,

L.H.S = $u_1 = 2$

R.H.S = $1^2 - 1 + 2 = 2$ (05)

$\therefore$ The result in true for n = 1

Take any $k \in \mathbb{Z}^+$ and assume that the result is true for n = k.

$u_k = k^2 - k + 2$ (05)

$u_{k+1} = u_k + 2k$ (05)

$= k^2 - k + 2 + 2k = k^2 + k + 2$

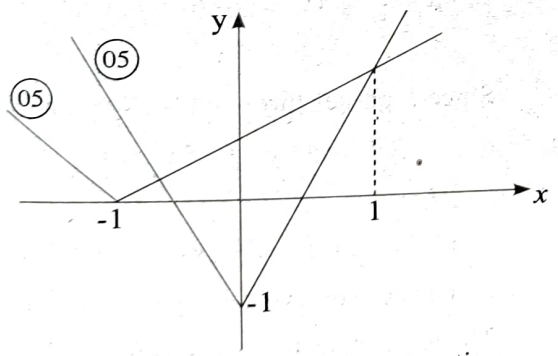
$= (k + 1)^2 - (k + 1) + 2$ (05)

Hence, if the result is true for n = k, Then it is also true for n = 1.

Hence, by the principal of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$. (05)

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02.



For the points of intersection,

$\frac{1}{2}(x + 1) = 2x - 1$ and $\frac{1}{2}(x + 1) = -2x - 1$

$\therefore x + 1 = 4x - 2$ and $x + 1 = -4x - 2$

$\therefore x = 1$ or $x = -\frac{3}{5}$ (05) For Both.

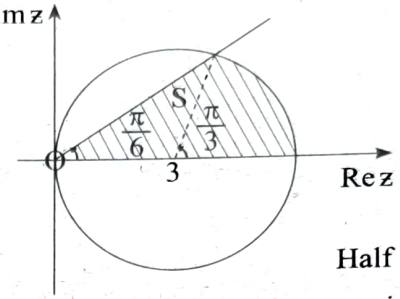
$4|x| - |x + 1| \leq 2$

$\Leftrightarrow |2x| - 1 \leq \frac{1}{2}|x + 1|$ (05)

$\Leftrightarrow -\frac{3}{5} \leq x \leq 1$ (05)

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03. Im z



Half Line (05)

Circle touching the y - a*x* is at O (05)

Region S with hole at O (05)

The area of S = $\underbrace{\left(\frac{1}{2} \times 3 \times 3 \times \frac{\sqrt{3}}{2}\right)}_{(05)} + \underbrace{\left(\frac{1}{2} \times 9 \times \frac{\pi}{3}\right)}_{(05)}$

$= \frac{9\sqrt{3}}{4} + \frac{3\pi}{2}$

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04. $\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{x}}\right)^{10}$

The general term = ${}^{10}C_r \left(\frac{1}{\sqrt{x}}\right)^r \left(\frac{1}{\sqrt{3}}\right)^{10-r}$

$= {}^{10}C_r \frac{1}{3^{10-r} x^{-r/2}}$ (05)

r = 6

The coefficient of $x^{-3} = {}^{10}C_6 \frac{1}{3^2} = A$ (say) (05)

r = 4

The coefficient of $x^{-2} = {}^{10}C_4 \frac{1}{3^3} = B$ (say) (05)

Since ${}^{10}C_6 = {}^{10}C_4$, we have 3B = A

(05) (05)

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05. $\lim_{x \to 1} \frac{1 + \cos \pi x}{(x - 1)^2}$

$= \lim_{x \to 1} \frac{1 + \cos \pi x}{(x - 1)^2} \frac{1 - \cos \pi x}{(1 - \cos \pi x)}$ (05)

$$= \lim_{x \to 1} \frac{\sin^2 \pi x}{(x-1)^2} \cdot \frac{1}{(1-\cos \pi x)} \quad (05)$$

$$= \lim_{x \to 1} \pi^2 \frac{\sin^2 \pi (x-1)}{\pi^2 (x-1)^2} \cdot \frac{1}{(1-\cos \pi x)} \quad (05)$$

$$= \pi^2 \cdot 1 \cdot \frac{1}{2} \quad (05) + (05)$$

$$= \frac{\pi^2}{2} \quad \boxed{25}$$

Aliter

$$\lim_{x \to 1} \frac{1+\cos \pi x}{(x-1)^2}$$

$$= \lim_{x \to 1} \frac{2\cos^2\left(\frac{x\pi}{2}\right)}{(x-1)^2}, \text{ where } u = x - 1 \quad (05)$$

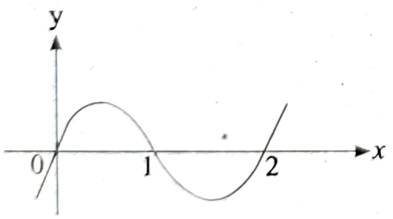
$$= \lim_{u \to 0} \frac{2\cos^2\left(\frac{\pi}{2}(u+1)\right)}{u^2} \quad (05)$$

$$= \lim_{u \to 0} \frac{2\sin^2 \frac{\pi u}{2}}{u^2} \quad (05)$$

$$= \lim_{u \to 0} 2 \left[\frac{\sin\left(\frac{\pi u}{2}\right)}{\frac{\pi u}{2}}\right]^2 \frac{\pi^2}{4} \quad (05)$$

$$= 2 \times 1^2 \times \frac{\pi^2}{4} = \frac{\pi^2}{2} \quad (05) \quad \boxed{25}$$

06.



$$\int_0^2 |f(x)|\,dx = \int_0^1 x(x-1)(x-2)\,dx + \int_1^2 -x(x-1)(x-2)\,dx$$

(05) (05)

$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx - \int_1^2 (x^3 - 3x^2 + 2x)\,dx \quad (05)$$

$$= \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2}\right]_0^1 - \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2}\right]_1^2 \quad (05)$$

$$= \left(\frac{1}{4} - 1 + 1\right) - \left[(4 - 8 + 4)\left(\frac{1}{4} - 1 + 1\right)\right] \quad (05)$$

$$= \frac{1}{2} \quad \boxed{25}$$

07. $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$

$$\frac{dx}{d\theta} = a - a\cos\theta \text{ and } \frac{dy}{d\theta} = a\sin\theta \quad (05) \quad (05)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a\sin\theta}{a(1 - \cos\theta)}$$

$$= \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2}$$

$$= \cot\theta/2 \quad (05)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cot\theta/2)$$

$$= \frac{d}{d\theta} \times (\cot\theta/2) \frac{d\theta}{dx} \quad (05)$$

$$= \frac{1}{2} \operatorname{cosec}^2\theta/2 \cdot \frac{1}{a(1 - \cos\theta)} < 0 \text{ for } 0 < \theta < \pi$$

C is concave downwards (05) $\boxed{25}$

08. The equation of l_3 has the form

$2y - x - 2 + \lambda(y + 2x - 6) = 0$, where $\lambda \in \mathbb{R}$ (05)

Since l_3 passes through $(-4, 4)$ we have

$8 + 4 - 2 + \lambda(4 - 8 - 6) = 0$ (05)

$\therefore \lambda = 1$

$\therefore l_3 : 2y - x - 2 + y + 2x - 6 = 0$ (05)

$\therefore$ The equation of l_3 is $3y + x - 8 = 0$

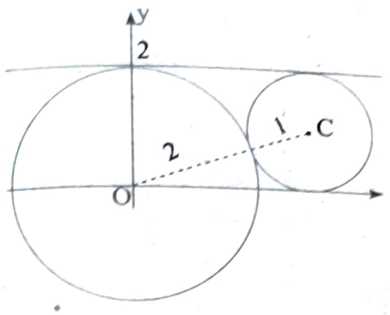
The angle bisectors are given by

$$\frac{2y - x - 2}{\sqrt{5}} = \pm \frac{y + 2x - 6}{\sqrt{5}} \quad (05)$$

(-) sign gives $3y + x - 8 = 0$ (05)

$\therefore l_3$ is an angle bisector of an angle between l_1 and l_2 $\boxed{25}$

09.



The centre of the required circle $C \equiv (x_0, 1)$ with $x_0 \in \mathbb{R}$ (05) and its radius = 1 (05)

since the two circles touch externally,

$OC = 2 + 1$

$= 3$ (05)

since $OC^2 = x_0^2 + 1$, we have $x_0^2 + 1 = 9$

$\therefore x_0 = \pm 2\sqrt{2}$ (05)

The required equations are

$(x - 2\sqrt{2})^2 + (y - 1)^2 = 1$ and

$(x + 2\sqrt{2})^2 + (y - 1)^2 = 1$ (05)

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10. $\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x + \frac{\pi}{4}\right) = \cos\frac{\pi}{24}$

$\Leftrightarrow 2\sin\left(x + \frac{7\pi}{24}\right)\cos\frac{\pi}{24} = \cos\frac{\pi}{24}$ (10)

$\Leftrightarrow \sin\left(x + \frac{7\pi}{24}\right) = \frac{1}{2}$ (05)

$\Leftrightarrow \sin\left(x + \frac{7\pi}{24}\right) = \sin\frac{\pi}{6}$ (05)

$x + \frac{7\pi}{24} = n\pi + (-1)^n \frac{\pi}{6}$ $n \in \mathbb{Z}$

$\Leftrightarrow x = n\pi + (-1)^n \frac{\pi}{6} - \frac{7\pi}{24}$, where $n \in \mathbb{Z}$

(05)

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Part B

11. (a) $f(x) = ax^2 + bx + c$

$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right\}$ (05)

$\geq -\frac{\Delta}{4a}$ (10) and = sign occurs when

$x = \frac{b}{2a}$ (05)

$\therefore$ The minimum value of $f(x) = -\frac{\Delta}{4a}$

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$g(x) = px^2 + 2\sqrt{pqx} + qr$

since,

$g(x) = 0$ has no real roots, its discriminant < 0 (05)

$\therefore 4pq - 4pqr < 0$ (05)

$4pq(1 - r) < 0$

since $4pq > 0$ we have $r > 1$ (05)

15

The minimum of $g(x) = \frac{-4pq(1 - r)}{4p}$ (05)

$= -q(1 - r)$

It is given that $-q(1 - r) = q$ (05)

$\therefore r = 2$ (05)

15

$g(x) = px^2 + 2\sqrt{pqx} + 2q$

since (0 , 1) is on the curve,

$1 = 2q \quad \therefore q = \frac{1}{2}$ (05)

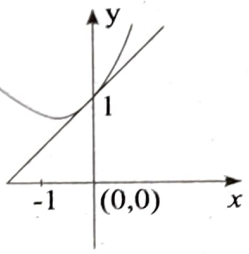
$g'(x) = 2px + 2\sqrt{pq}$ (05)

$g'(0) = 2\sqrt{pq}$ (05)

$g'(0) = 1$ (05)

$\therefore 2\sqrt{pq} = 1$ (05)

$4pq = 1 \quad p = \frac{1}{2}$ (05)



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(b) $P(x) = (x - a)\,q(x)$, where $q(x)$ is a polynomical of degree 3 (05)

$P'(x) = (x - a)\,q'(x) + q(x)$ —— (1) (05)

since $(x - a)$ is a factor of $p'(x)$ also

$p'(a) = 0$ (05)

(1) $\Rightarrow g(a) = 0$ (05)

$\therefore (x - a)$ is a factor of $q(x)$ (05)

$\therefore q(x) = (x - a)\,r(x)$, when $r(x)$ is a polynomial of degree 2.

Then $p(x) = (x - a)^2\,r(x)$ (05)

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$f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

$f(2) = 16 - 8 - 24 + 8 + 8 = 0$ (05)

$\therefore$ $(x - 2)$ is a factor of $f(x)$ (05)

$f'(x) = 4x^3 - 3x^2 - 12x + 4$

$f'(2) = 32 - 12 - 24 + 4 = 0$ (05)

$\therefore$ $(x - 2)$ is a factor of $f'(x)$ (05)

$\therefore$ $(x - 2)^2$ is a factor of $f(x)$ (05)

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$f(-1) = 1 + 1 - 6 - 4 + 8 = 0$ (05)

$\therefore$ $(x + 1)$ is a factor of $f(x)$

So $f(x) = (x - 2)^2 (x + 1)(x + k)$, where $k \in \mathbb{R}$ (05)

Now $f(0) = 8 = 4k$

$k = 2$

$\therefore f(x) = (x - 2)^2 (x + 1)(x + 2)$ (05)

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12. (a) (i) ${}^8C_4 \times {}^6C_4$ (10)

(10)

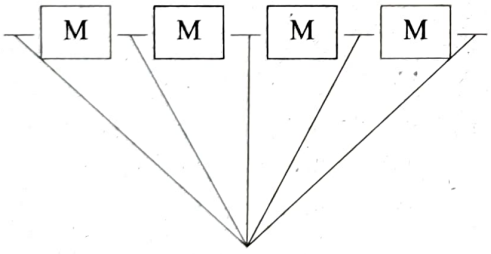
$= \frac{8!}{4!\,4!} \times \frac{6!}{2!\,4!}$ (05)

$\frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2} \times \frac{6 \times 5}{2}$

$= 70 \times 15 = 1050$ (05)

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(ii)



The number of ways of choosing 4 out of these 5 places

$= {}^5C_4$ (05)

The Answer $= {}^5C_4 \times 4! \times 4!$ (05) + (05)

(05)

$= 2880$ (05)

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(b)

$\sum_{r=1}^{n} u_r = \frac{n}{4}(n + 1)(n + 2)(n + 3)$

$\sum_{r=1}^{n-1} u_r = \frac{(n-1)}{4} n(n+1)(n+2)$ for $n \geq 2$ (05)

$u_1 = \frac{1}{4}\, 2.\, 3.\, 4. = 6$ (05)

$\therefore u_n = \sum_{r=1}^{n} u_r - \sum_{r=1}^{n-1} u_r$ for $n \geq 2$

$= \frac{n}{4}(n + 1)(n + 2)(n + 3) - \frac{(n-1)}{4} n(n + 1)(n + 2)$ (05)

$= \frac{n}{4}(n + 1)(n + 2)[(n + 3) - (n - 1)]$

$= n(n + 1)(n + 2)$ (05)

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Now, $V_r = \frac{1}{r(r + 1)(r + 2)}$ for all $r \in \mathbb{Z}^+$

$\frac{1}{r(r + 1)(r + 2)} = \frac{A}{r(r + 1)} + \frac{B}{(r + 1)(r + 2)}$ (05)

$1 = A(r + 2) + Br$, for all $r \in \mathbb{Z}^+$

$1 = (A + B)r + 2A$, for all $r \in \mathbb{Z}^+$

$r^1 : A + B = 0$ (05) $A = \frac{1}{2}$ $B = -\frac{1}{2}$

$r^0 : 1 = 2A$ (05) (05)

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$V_r = \frac{\frac{1}{2}}{r(r + 1)} - \frac{\frac{1}{2}}{(r + 1)(r + 2)}$

$r = 1$ $V_1 = \frac{1}{2}\left[\frac{1}{1.2} - \frac{1}{2.3}\right]$ (05)

$r = 1$ $V_2 = \frac{1}{2}\left[\frac{1}{2.3} - \frac{1}{3.4}\right]$

$r = n - 1$ $V_{n-1} = \frac{1}{2}\left[\frac{1}{(n-1)n} - \frac{1}{n(n+1)}\right]$

(05)

$r = n$ $V_n = \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]$

(10)

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$$\sum_{r=1}^{n} V_r = \frac{1}{2}\cdot\frac{1}{12} - \frac{1}{2}\frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

$\to \frac{1}{4}$ as $n \to \infty$ (05)

$\sum_{r=1}^{\infty} V_r$ is convergent and its sum $= \frac{1}{4}$

(05) (05)

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$$\sum_{r=m}^{\infty} V_r = \sum_{r=1}^{\infty} V_r - \sum_{r=1}^{m-1} V_r \quad (05)$$

$$= \frac{1}{4} - \left\{\frac{1}{4} - \frac{1}{2m(m+1)}\right\}$$

$$= \frac{1}{2m(m+1)} \quad (05)$$

$\therefore \frac{1}{2m(m+1)} = \frac{1}{24} \Leftrightarrow m^2 + m - 12 = 0$ (05)

$\Leftrightarrow (m-3)(m+4) = 0$

$\Leftrightarrow m = 3, m = -4$

$m = 3$ (since $m \in \mathbb{Z}^+$)

(05) 20

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13. (a) (05)

$$\begin{vmatrix} 1 & a \\ -a & 1 \end{vmatrix} = 1 + a^2 \neq 0 \ (\because m \in \mathbb{R}) \quad (05)$$

$$A^{-1} = \frac{1}{1+a^2}\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} \quad (05) \ (10)$$

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(i) $A^{-1}B^T = \frac{1}{1+a^2}\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (05)

$$= \frac{1}{1+a^2}\begin{pmatrix} 1-a & -1-a \\ a+1 & -a+1 \end{pmatrix} \quad (05)$$

since this is given to be equal to $-\frac{1}{5}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$ we have

(05) (05)

$\frac{1-a}{1+a^2} = -\frac{1}{5}$ and $\frac{-1-a}{1+a^2} = -\frac{3}{5}$

$\Leftrightarrow 1 + a^2 = -5 + 5a$ and $3 + 3a^2 = 5 + 5a$

$\Leftrightarrow a^2 - 5a + 6 = 0$ and $3a^2 - 5a - 2 = 0$

$\Leftrightarrow (a-2)(a-3) = 0$ and $(a-2)(3a+1) = 0$ (05)

$\Leftrightarrow$ ($a - 2$ or $a = 3$) and ($a = 2$ or $a = \frac{-1}{3}$)

$a = 2$ (05)

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(ii) $C = B^{-1}\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 5 \end{bmatrix}$ (05)

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 5 \end{pmatrix} \quad (05)$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 2 & -2 \\ 2 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 4 \end{pmatrix} \quad (05)$$

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(b) Let $z = x + iy$, where $x, y \in \mathbb{R}$

$\bar{z} = x - iy$ and $|z| = \sqrt{x^2 + y^2}$

(05) (05) 10

Let $|z| = 1$

Then $z\bar{z} = |z|^2 = 1$ (05)

$\bar{z} = \frac{1}{z}$ (05)

$|1 - \bar{z}w| = |1 - \frac{1}{z}w| = \frac{1}{|z|}|z - w| = |z - w|$ (10)

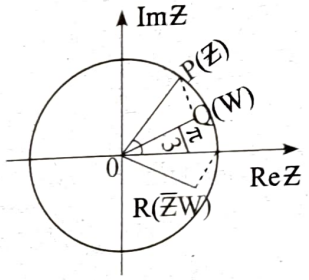
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$z = \frac{1}{2}(1 + \sqrt{3}i)$

$|z| = \frac{1}{2}\sqrt{1+3} = 1$ (05)

$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ (05)

$\therefore \text{Arg } z = \frac{\pi}{3}$ (05) 15



Δ OPG $\equiv \Delta$ OAR (05)

$\therefore$ AB = CD

$\therefore |z - w| = |1 - \bar{z}w|$ (05) 10

(c) $\dfrac{\left(\cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}\right)^n}{\left(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}\right)^7}$

$= \dfrac{\left(\cos\frac{2n\pi}{15} + i\sin\frac{2n\pi}{15}\right)}{\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)}$ (05) (05)

$= \cos\left[\frac{\pi}{15}(2n-7)\right] + i\cos\left[\frac{\pi}{15}(2n-7)\right]$ (05)

$\therefore \cos\left[\frac{\pi}{15}(2n-7)\right] = \frac{1}{2}$ (05)

$n = 1$ (05) 25

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14. (a) $P = -4$ and $q = 1$ (05) (05) 10

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{(ax+1)(x+2)}{(x+4)(x-1)}$

$= \lim_{x\to\infty} \dfrac{(a+\frac{1}{x})(1+\frac{2}{x})}{(1+\frac{4}{x})(1-\frac{1}{x})}$

$= a$ (05)

$\therefore a = 1$ (05) 10

$f(x) = \dfrac{(x+1)(x+2)}{(x+4)(x-1)}$ (05) (05)

$f'(x) = \dfrac{(x+4)(x-1)(2x+3) - (x+1)(x+2)(2x+3)}{(x+4)^2(x-1)^2}$ (05)

$= \dfrac{(2x+3)[(x^2+3x-4) - (x^2+3x+2)]}{(x+4)^2(x-1)^2}$

$= \dfrac{-6(2x+3)}{(x+4)^2(x-1)^2}$ (05)

$f'(x) = 0 \Leftrightarrow x = -\frac{3}{2}$ (05)

	sign of $f'(x)$ $\dfrac{-6(2x+3)}{(x+4)^2(x-1)^2}$	$f(x)$
$-\infty < x < -4$	$\frac{(-)(-)}{(+)} = (+)$ (05)	↗
$-4 < x < -\frac{3}{2}$	$\frac{(-)(-)}{(+)} = (+)$ (05)	↗
$-\frac{3}{2} < x < 1$	$\frac{(-)(+)}{(+)} = (-)$ (05)	↘
$1 < x < \infty$	$\frac{(-)(+)}{(+)} = (-)$ (05)	↘

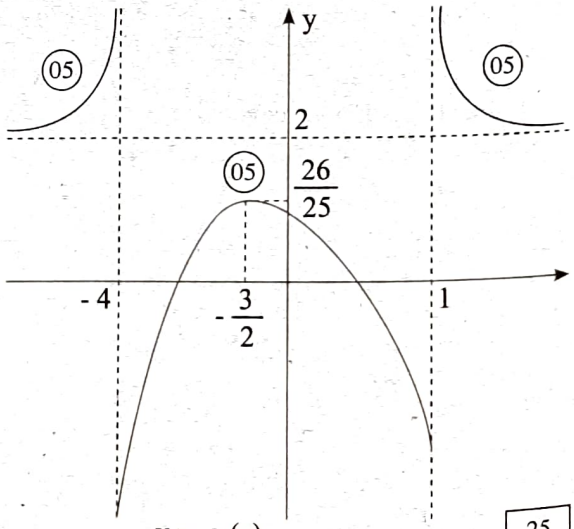
Increasing on : $(-\infty, -4)$ and $(-4, -\frac{3}{2})$ (05)

Decreasing on : $(-\frac{3}{2}, 1)$ and $(1, \infty)$ (05) 55

$g(x) = f(x) + 1$ Turning point $(-\frac{3}{2}, \frac{26}{25})$ (05)

$g'(x) = f'(x)$ $g(-\frac{3}{2}) = \frac{26}{25}$ (05)

$g(0) = \frac{1}{2}$

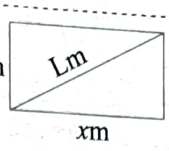


$y = g(x)$ 25

The range of g (x) : $\left(-\infty, \frac{26}{25}\right] \cup (2, \infty)$ 10

(b) $xy = k$

$y = \frac{k}{x}$ (05)



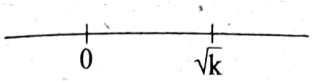
$L^2 = x^2 + y^2$ (05)

$\Rightarrow L^2 = x^2 + \frac{k^2}{x^2}$ (05)

$2L \frac{dL}{dx} = 2x - \frac{k^2}{x^3}$ (10)

$\frac{dL}{dx} = 0 \Leftrightarrow x^4 = k^2$

(05). $\Leftrightarrow x = \sqrt{k}$ (05)



$0 < x < \sqrt{k}$, $\frac{dL}{dx} < 0$

$\sqrt{k} < x < \infty$, $\frac{dL}{dx} > 0$

$\therefore$ L is minimum when $x = \sqrt{k}$ (05) 40

15. (a) When $k \neq 0$

$\frac{1}{x^2(x-k)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-k}$ (10)

$1 = Ax(x-k) + B(x-k) + Cx^2$ (05)

$1 = (A + C)x^2 + (B - A)x - Bk$

By comparing the coefficients

x^0 , $1 = -Bk$ (05)

x^1 , $0 = B - Ak$ (05)

x^2 , $0 = A + C$ (05)

$B = -\frac{1}{k}$, $A = -\frac{1}{k^2}$, $C = \frac{1}{k^2}$

$\int \frac{1}{x^2(x-k)} dx$

$= -\frac{1}{k^2}\int \frac{1}{x} dx - \frac{1}{k}\int \frac{1}{x^2} dx + \frac{1}{k^2}\int \frac{1}{(x-k)} dx$ (05)

$\int \frac{1}{x^2(x-k)} dx$

$= -\frac{1}{k^2}\int \ln|x| + \frac{1}{kx} + \frac{1}{k^2}\ln|x-k| + D$

Where D is an arbitrary constant. (15)

When $k = 0$

$\int \frac{1}{x^3} dx = \frac{x^{-2}}{-2} + C = \frac{1}{2x^2} + D$; (05)

Where D is an arbitrary constant. 55

(b) $\int_1^{e^{\frac{\pi}{2}}} x \sin(\ln x)\, dx$

$= \left[\frac{x^2}{2}\ln|x|\right] - \int_1^{e^{\frac{\pi}{2}}} \frac{x^2}{2}\cos\ln|x| \frac{1}{x} dx$ (15)

$= \left[\frac{x^2}{2}\ln|x|\right] - \frac{1}{2}\int_1^{e^{\frac{\pi}{2}}} x\cos\ln|x|\, dx$ (05)

$\frac{e^{\pi}}{2} = \int_1^{e^{\frac{\pi}{2}}} x\sin(\ln x)\, dx + \frac{1}{2}\int_1^{e^{\frac{\pi}{2}}} x\cos(\ln x)\, dx$ (10)

$\int_1^{e^{\frac{\pi}{2}}} x\left[2\sin\ln|x|\right] + \left\{\cos\ln|x|\right\} dx = e^{\pi}$ (10)

40

(c)

$\frac{d}{dx}\left[(k\sqrt{x} - 1)e^{k\sqrt{x}}\right]$

$= \frac{1}{2}(k\sqrt{x} - 1)e^{k\sqrt{x}} kx^{-\frac{1}{2}} + \frac{1}{2}e^{k\sqrt{x}} kx^{-\frac{1}{2}}$ (10)

$= \frac{k^2}{2}e^{k\sqrt{x}}$ (05) 15

$I_k = \int_1^4 e^{k\sqrt{x}}\, dx = \frac{2}{k^2}\left[(k\sqrt{x} - 1)e^{k\sqrt{x}}\right]$ (10)

$I_k = \frac{2}{k^2}\left\{(2k - 1)e^{2k} - (k - 1)e^{k}\right\}$ (05)

15

The area of S $= \int_1^4 e^{\sqrt{x}}\, dx$ (05)

$= I_1$

$= 2\left\{(2 - 1)e^2\right\}$

$= 2e^2$ (05) 10

The required volume $= \pi \int_1^4 e^{2\sqrt{x}}\, dx$ (10)

$= \pi I_2$

$= \frac{\pi}{2}\{(4-1)e^4-(2-1)e^2\}$

$= \frac{\pi}{2}\{3e^4 - e^2\}$ (05)

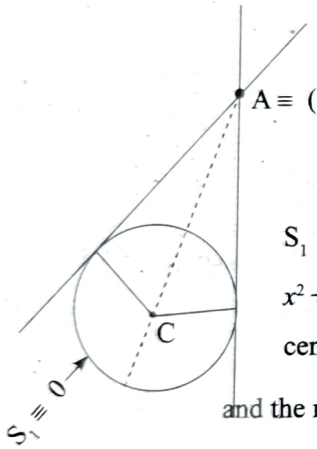
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16. l : $y - 1 = m(x - 3)$

i. e. $mx - y + 1 - 3m = 0$ (10)

10



$S_1 = 0$ is the same as

$x^2 + y^2 - 2x + 2y + \frac{6}{5} = 0$

centre of S_1 = O is (1, -1) (05)

and the radius $= \sqrt{2 - \frac{6}{5}} = \frac{2}{\sqrt{5}}$ (05)

The line l is a tan gent to $S_1 = 0$ if and only if (05)

$\frac{|-1-m+3m-1|}{\sqrt{1+m^2}} = \frac{2}{\sqrt{5}}$ (05) (05)

$\Leftrightarrow \frac{|2m-2|}{\sqrt{1+m^2}} = \frac{2}{\sqrt{5}}$

$\Leftrightarrow \frac{|m-1|}{\sqrt{1+m^2}} = \frac{1}{\sqrt{5}}$

$\Leftrightarrow 5(m^2 - 2m + 1) = 1 + m^2$ (10)

$\Leftrightarrow 4m^2 - 10m + 4 = 0$

$\Leftrightarrow (2m - 1)(m - 2) = 0$ (05)

$m = \frac{1}{2}$, or $m = 2$ (10)

There are two tangent lines to the circle $S_1 = 0$ through A (05)

55

The two tangents are : $y - 2x + 5 = 0$ (05)

and $y - \frac{1}{2}x + \frac{1}{2} = 0$ (05)

Let $\propto$ be the angle between them

Then $\tan \propto$

$= \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{2 - \frac{1}{2}}{1 + 1} = \frac{3}{4}$ (05)

$\propto = \tan^{-1}\left(\frac{3}{4}\right)$ (05)

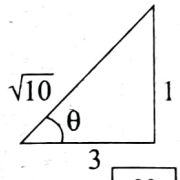
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Aliter

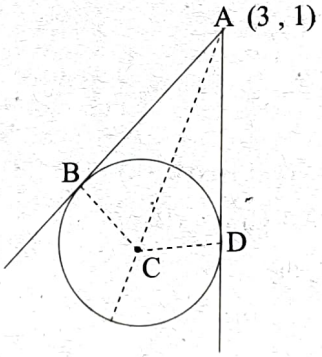
The distance between A and C $= \sqrt{2^2 + 2^2} = 2\sqrt{2}$ (05)

$\therefore \sin\theta = \frac{2\sqrt{5}}{2\sqrt{2}} = \frac{1}{\sqrt{10}}$ (10)

$\therefore \propto = 2\sin^{-1} = \left(\frac{1}{\sqrt{10}}\right)$ (05)



20



$A\hat{B}C = \frac{\pi}{2}$ and $A\hat{D}C = \frac{\pi}{2}$

$\therefore A\hat{B}C + A\hat{D}C = \pi$ (05)

$\therefore$ ABCD is a cyclio quadrilateral (05)

$\therefore$ AC is a diameter of the required circle.

$\therefore$ The required equation is (05)

$(x - 1)(x - 3) + (y + 1)(y - 1) = 0$

i.e. $x^2 + y^2 - 4x + 2 = 0$ (10)

25

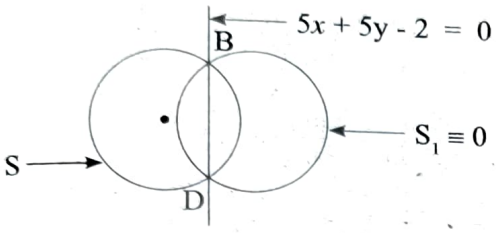
Aliter

Centre $\equiv (2, 0)$ (05) and the radius $= \frac{1}{2}\sqrt{8}$

$= \sqrt{2}$ (10)

$(x - 2)^2 + y^2 = 2$ (10)

$x^2 + y^2 - 4x + 2 = 0$ 25



The equation of BD :

$3x + y - (x + 3) + (y + 1) + \frac{6}{5} = 0$ (10)

i.e. $10x + 10y - 4 = 0$

i.e. $5x + 5y - 2 = 0$

Let S_2 be the requied circle for the last part

Then the center of S_2 is $A \equiv (3, 1)$ (10)

Also $C = (1, -1)$

$\therefore AC = \sqrt{(3 - 1)^2 + (-1 - 1)^2} = 2\sqrt{2}$ (05)

Let $AB = R$

$AC^2 = R^2 + BC^2$ (05)

$8 = R^2 + \frac{4}{5}$

$\therefore r_2 = \frac{6}{\sqrt{5}}$ (05)

$S_2 : (x - 3)^2 + (y - 1)^2 = \frac{36}{5}$

$5x^2 + 5y^2 - 30x - 10y + 14 = 0$ (05)

40

Aliter

The equation of a circle S_2 through B and D has the form $x^2 + y^2 - 2x + 2y - \frac{6}{4} + \lambda(5x + 5y - 2) = 0$, where $\lambda \in \mathbb{R}$ (10)

i.e. $x^2 + y^2 - (2 - 5\lambda)x + (2 + 5\lambda)y - \frac{6}{4} - 2\lambda = 0$ (05)

$S_2 \equiv 0$ inter sects $S_1 \equiv 0$ orthogonally

$\Leftrightarrow 2(-1)\left[\frac{-(2 - 5\lambda)}{2}\right] + 2(1)\left(\frac{2 + 5\lambda}{2}\right) = \frac{6}{5} + \frac{6}{5} - 2\lambda$ (10)

$\Leftrightarrow 2 - 5\lambda + 2 + 5\lambda = \frac{12}{5} - 2\lambda$ (05)

$2\lambda = -\frac{8}{5}$

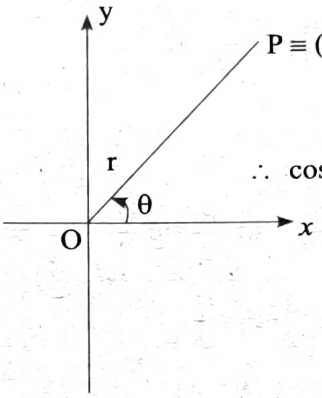
$\lambda = -\frac{4}{5}$ (05)

$\therefore$ The equation of S_2 is $x^2 + y^2 - 6x - 2y + \frac{14}{5} = 0$ (05)

40

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17. (a) $r = \sqrt{x^2 + y^2}$



$\cos\theta = \frac{x}{r}$ (05)

$\sin\theta = \frac{y}{r}$ (05)

$\therefore \cos^2\theta + \sin^2\theta = \frac{x^2}{r^2} + \frac{y^2}{r^2}$ (05)

$= \frac{x^2 + y^2}{r^2}$

$= \frac{x^2}{r^2}$

$= 1$ (05)

20

$\cos^2 x - 1 = \sin^2 x + 3\cos x$

$\cos^2 x - 1 = 1 - \cos^2 x + 3\cos x$ (05)

$2\cos^2 x - 3\cos^2 x - 2 = 0$

$(2\cos x + 1)(\cos x - 2) = 0$ (05)

$\cos x = -\frac{1}{2}$ (05) $\cos x \neq 2$

$\cos x = \cos\frac{2\pi}{3}$

$x = 2n\pi \pm \frac{2\pi}{3}$ for $n \in \mathbb{Z}$ (10)

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$ (05) 30

(b) $\sin\left(\frac{B + C}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{A}{2}\right) = \cos\frac{A}{2}$ (05) (05)

$\cos\left(\frac{B + C}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin\frac{A}{2}$ (05) (05)

$A + B + C = \pi$

$\frac{B + C}{2} = \frac{\pi}{2} - \frac{A}{2}$ 20

$$\tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) =$$

$$\frac{\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} \quad (05)$$

$$= \frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}} \quad (05)$$

$$= \cos\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2} \quad (05)$$

$$1 - \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$$

$$= \frac{\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} \quad (05)$$

$$1 - \tan\frac{A}{2}\tan\frac{C}{2} = \frac{\cos\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)} \quad (05)$$

$$= \sin\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2} \quad (05)$$

30

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2}$$

$$= \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2}$$

$$= \tan\frac{A}{2}\cos\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2}$$

(05)

$$= \sin\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2} \quad (05)$$

$$= 1 - \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2}$$

$$= 1 \quad (05)$$

15

(c) $\underbrace{\tan^{-1}(2x)}_{\alpha} + \underbrace{\tan^{-1}(3x)}_{\beta} = \frac{3\pi}{4}$

$\tan\alpha = 2x$ and $\tan\beta = 3x$ (05)

$\alpha + \beta = \frac{3\pi}{4}$

$\tan(\alpha + \beta) = \tan\frac{3\pi}{4}$ (05)

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan\frac{3\pi}{4} \quad (05)$$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 1$$

$$\frac{2x + 3x}{1 - 6x^2} = -1 \quad (05)$$

$6x^2 - 5x - 1 = 0$

$(6x + 1)(x - 1) = 0$ (05)

$x = -\frac{1}{6}$ or $x = 1$ (05)

$x = -\frac{1}{6}$ does not satisfy the equation

$x = 1$ satisfies the equation.

$\therefore x = 1$ (05)

35
