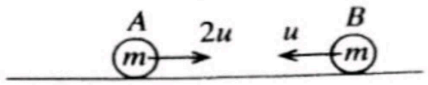
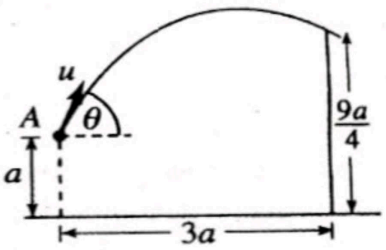
Part A

1. Two particles A and B, each of mass m, moving on a smooth horizontal table along the same straight line towards each other with speeds $2u$ and u, respectively, collide directly.

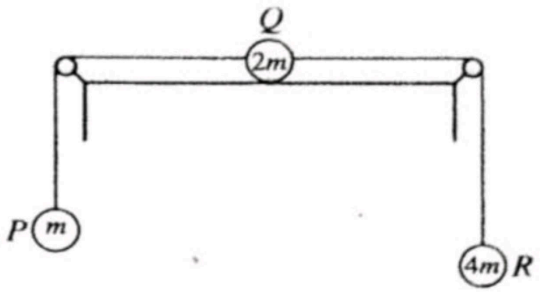


The coefficient of restitution between A and B is e. Find the velocities of A and B after the collision, and show that if $e=\frac{1}{3}$, then A will come to rest after the collision.

2. A particle is projected from a point A at a distance a above a horizontal floor at an angle $\theta\left(0<\theta<\frac{\pi}{2}\right)$ to the horizontal with an initial velocity of $u=3\sqrt{\frac{ga}{2}}$. It just clears a vertical wall of height $\frac{9a}{4}$ situated at a horizontal distance $3a$ from A. (See the figure.) Show that $\theta=\tan^{-1}\left(\frac{3}{2}\right)$.



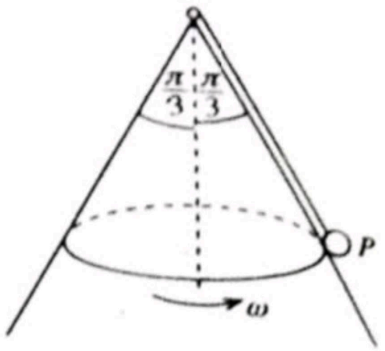
3. Three particles P, Q and R of masses m, $2m$ and $4m$, respectively are connected by two light inextensible strings passing over two small smooth pulleys fixed at two edges of a smooth horizontal table, as shown in the figure. The system is released from rest with strings taut. Obtain equations sufficient to determine the acceleration of R.



4. A van of mass 1000 kg is travelling along a straight horizontal road against a constant resistance of magnitude 800 N. Find the power of the engine of the van when it is travelling with a constant speed of 50 m s^{-1}.

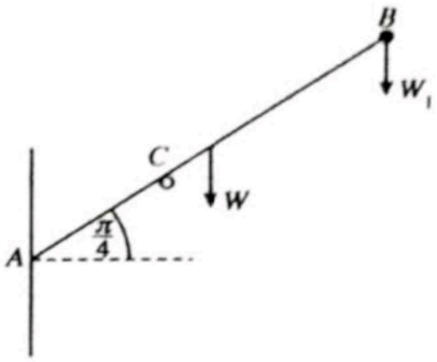
Now, the van travels along another straight horizontal road against a resistance of $(800 + 4v)$ N, where v m s^{-1} is the speed of the van. Find the acceleration of the van at the instant when its speed is 20 m s^{-1}, if the engine of the van is working at the same power.

5. A particle P of mass m is attached to the vertex of a fixed smooth right-circular cone by a light inextensible string of length l. The semi-vertical angle of the cone is $\frac{\pi}{3}$ and the axis of the cone is vertical. (See the figure.) The particle P is moving on the surface of the cone in a horizontal circle with an angular speed $\omega = \sqrt{\frac{2g}{l}}$. Find the tension in the string.

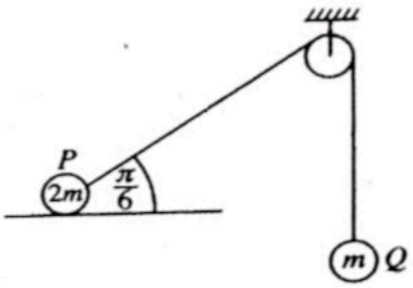


6. In the usual notation, with respect to a fixed origin O let the position vectors of two points A and B, be $2\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} - 4\mathbf{j}$, respectively. Let C be the point such that $2\overrightarrow{OA} + \overrightarrow{BC} = 2\overrightarrow{OC}$. Find $\overrightarrow{OC}$ in terms of $\mathbf{i}$ and $\mathbf{j}$ and, find the unit vector in the direction of $\overrightarrow{OC}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

7. A uniform rod AB of length $4a$ and weight W is kept with its end A against a smooth vertical wall. The rod is placed on a peg at the point C on the rod such that $AC = \frac{3a}{2}$. A particle of weight W_1 is fixed at the other end B of the rod. The rod makes an angle of $\frac{\pi}{4}$ with the horizontal, as shown in the figure. The rod is in equilibrium. Show that $W_1 = W$.



8. A particle P of mass $2m$ is on a rough horizontal table. One end of a light inextensible string passing over a fixed smooth pulley is attached to the particle P and the other end is attached to a particle Q of mass m, as shown in the diagram. The section of the string from P to the pulley, makes an angle $\frac{\pi}{6}$ with the table. The system is in equilibrium with the string taut. Show that the coefficient of friction μ between the particle P and the table satisfies $\mu \geq \frac{1}{\sqrt{3}}$.



9. Let A and B be two events of a sample space Ω. It is given that $P(A \cap B) = \frac{1}{3}$ and $P(A' \cap B) = \frac{1}{12}$, where A' denotes the complementary event of A.
Find $P(B)$.
Now, it is also given that A and B are independent events. Find $P(A \cup B)$.

10. The mean of the set of data $\{x_1, x_2, \ldots, x_{10}\}$ is 30 and $\sum_{i=1}^{10}(x_i - 20)^2 = 1040$.
Find the variance of the above set of data.

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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka
අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2023 (2024)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2023 (2024)
General Certificate of Education (Adv. Level) Examination, 2023 (2024)

සංයුක්ත ගණිතය II
இணைந்த கணிதம் II
Combined Mathematics II

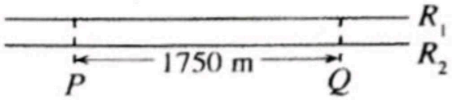
10 E II

Part B

* Answer **five** questions only.

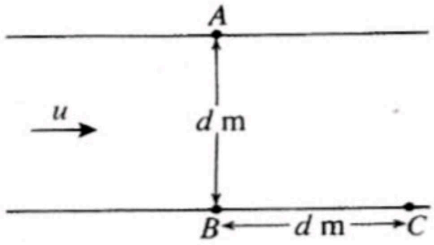
(In this question paper, g denotes the acceleration due to gravity.)

11. (a) R_1 and R_2 are two straight parallel train tracks between two railway stations P and Q that are 1750 m apart. A train A that starts from rest at $t = 0$ from the station P travels along the track R_1 with a uniform acceleration of 10 m s^{-2} for T seconds and then maintains the speed it gains at $t = T$ s for 30 seconds. It then decelerates uniformly for T seconds and comes to rest at the station Q. Sketch the velocity-time graph for the motion of the train A from P to Q, and show that the total time taken for the journey is 40 s.



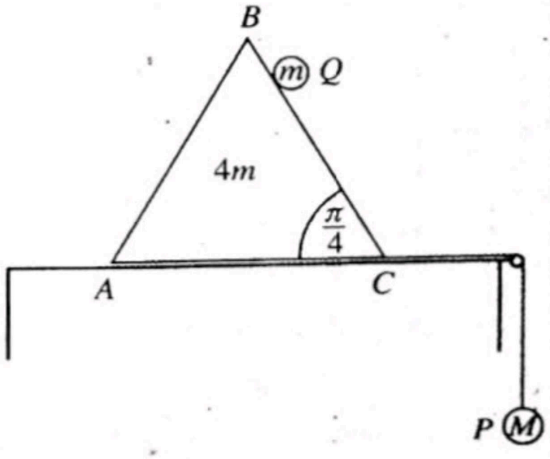
Another train B travelling along the track R_2 in the direction of $\overrightarrow{PQ}$ at a constant speed of 40 m s^{-1} passes the station P at $t = 0$. Sketch the velocity-time graph for the motion of train A relative to the train B from $t = 0$ to $t = 40$ s.

(b) A river d m wide, flows at a uniform speed u m s^{-1} in between two straight parallel banks. A swimmer P whose speed relative to water is $\sqrt{2}\,u$ m s^{-1}, starts from a point A on one bank and swims to reach the point B on the other bank directly opposite to A. Show that the time taken by the swimmer P to reach B is $\frac{d}{u}$ s.



A second swimmer Q whose speed relative to water is $2\sqrt{2}\,u$ m s^{-1}, starts from a point C further down the river, d m from B on the same bank, and swims with the intention of meeting the swimmer P. (See the figure.) Assuming that the swimmers P and Q start to swim at the same instant, show that the swimmer Q meets the swimmer P before the swimmer P reaches the point B.

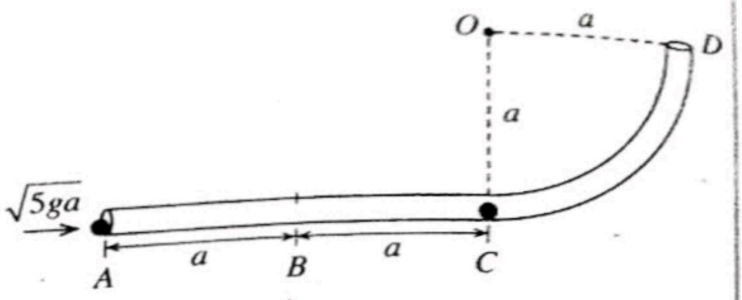
12. (a) The vertical cross-section ABC through the centre of gravity of a smooth uniform wedge of mass $4m$ is shown in the figure. The face containing AC is placed on a smooth horizontal table. Also, AB and BC are lines of greatest slope of the faces containing them and $A\hat{C}B = \frac{\pi}{4}$. The point C of the wedge and a particle P of mass M are attached to the ends of a light inextensible string passing over a smooth small pulley fixed to an edge of the table. The string lies in the same vertical plane containing ABC. A particle Q of mass m is held on BC. The particle P hangs freely. The system is released from rest from this position with the string taut.



Show that particle P moves vertically downwards, if $m < 2M$.

Describe the motions of each of the particles and the wedge, if $m = 2M$.

(*b*) A thin tube $ABCD$ is fixed in a vertical plane with ABC horizontal, as shown in the figure. Each of the sections AB and BC is of length a, and the section CD is a quarter circle of radius a and centre O with OC vertical.



A particle P of mass m is placed inside the tube at the point C. Another particle Q of mass m is placed inside the tube at the point A, and it is given a velocity of magnitude $\sqrt{5ga}$ in the direction of $\overrightarrow{AB}$. The coefficient of friction between the particle Q and the section AB is $\frac{1}{2}$, and the section BCD is smooth.

The particle Q moves through the tube, and collides and coalesces with the particle P. Find the velocity with which this combined particle R starts to move.

Show that the speed v of the particle R, when $\overrightarrow{OR}$ turns through an angle θ with the downward vertical is given by $v^2 = ga(2\cos\theta - 1)$, and find the reaction on the particle R from the tube at the instant when it comes to instantaneous rest inside the tube.

13. A composite particle P of mass $2m$ is made by gluing together two particles each of mass m. One end of a light elastic string of natural length a and modulus of elasticity $2mg$ is attached to a fixed point O on a horizontal ceiling, and the other end to the composite particle P. The particle P is hanging in equilibrium at a point A. Find the extension of the string at this equilibrium position.



Show that if the particle P is pulled down a distance of $\frac{a}{2}$ from A and released, the equation of motion of P is $\ddot{x} + \omega^2 x = 0$ for $-\frac{a}{2} \le x \le \frac{a}{2}$, where $\omega = \sqrt{\frac{g}{a}}$ and $AP = x$.

Now, the particle P is pulled down a distance of l from A and released.

What is the maximum value of l for which the particle P will perform a complete simple harmonic motion?

Find the value of l for which the particle P will hit the point O with a speed of $\sqrt{ag}$.

When the particle P hits O at this speed, one particle of mass m falls off. The ceiling is inelastic. Obtain the equation of motion for the new simple harmonic motion that the remaining particle performs, after its motion under gravity.

Find the time taken from O for this single particle to come to instantaneous rest for the first time.

14. (*a*) In the usual notation, the position vectors of four points A, B, C and D are $\mathbf{a} = -\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 4\mathbf{j}$, $\mathbf{c} = 8\mathbf{i} + \alpha\mathbf{j}$ and $\mathbf{d} = 4\mathbf{i} - 2\mathbf{j}$, respectively, where $\alpha \in \mathbb{R}$.

The lines AB and DC are parallel. Show that $\alpha = 8$.

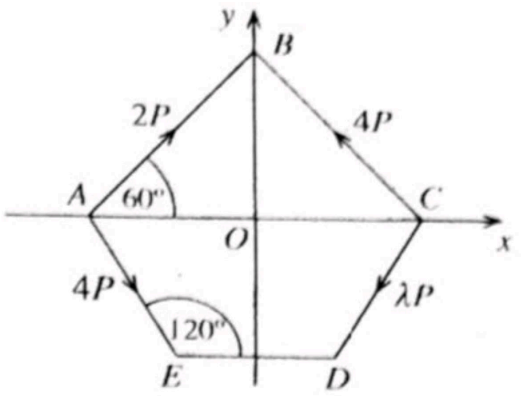
The lines AC and BD intersect at a point E with the position vector $\mathbf{e}$. By considering $\overrightarrow{AE}$ and $\overrightarrow{AC}$, show that $\mathbf{e} = (1-\lambda)\mathbf{a} + \lambda\mathbf{c}$ for $\lambda \in \mathbb{R}$.

Similarly, also show that $\mathbf{e} = (1-\mu)\mathbf{b} + \mu\mathbf{d}$ for $\mu \in \mathbb{R}$.

Hence find $\mathbf{e}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

By considering $\overrightarrow{EA} \cdot \overrightarrow{ED}$, find $A\hat{E}D$.

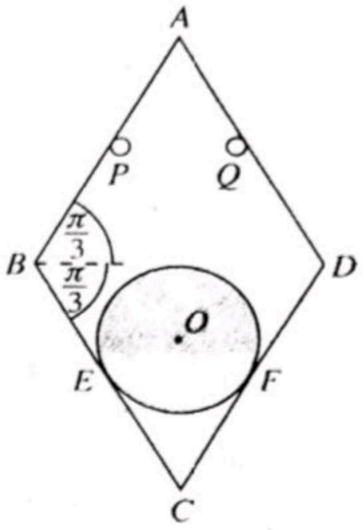
(b) The pentagon $ABCDE$ shown in the figure is symmetric about the y-axis. The vertices A and C lie on the x-axis and the vertex B lies on the y-axis. Also, $AC = 4a$, $DE = 2a$, $A\hat{E}D = 120^\circ$ and $O\hat{A}B = 60^\circ$, where O is the origin.



Four forces of magnitudes $2P$, $4P$, λP and $4P$ act along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$ and $\overrightarrow{AE}$, respectively, where $\lambda \in \mathbb{R}$. It is given that this system of forces is equivalent to a single force $\mathbf{R}$ acting through O. Find the value of λ and the magnitude and the direction of $\mathbf{R}$.

Now, a force of magnitude $2P$ acting along $\overrightarrow{DE}$ and a couple with moment $4\sqrt{3}Pa$ acting in the counterclockwise sense are added to the above system. Find the magnitude, the direction and the equation of the line of action of the single force to which the new system reduces.

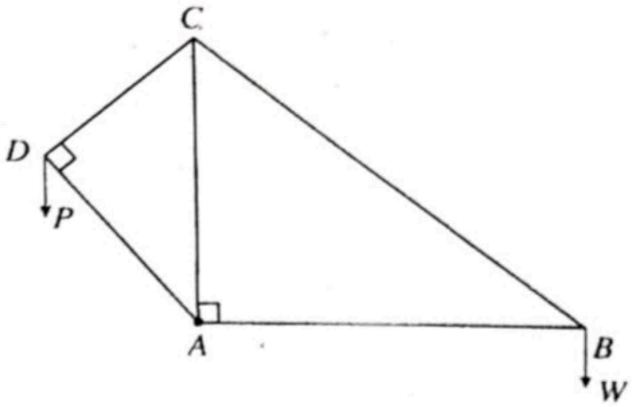
15.(a) Four uniform rods AB, BC, CD and DA of equal length $2a$ and equal weight W are smoothly jointed at the points A, B, C and D. A smooth uniform thin circular disk of centre O, radius $\frac{a}{\sqrt{3}}$ and weight W is placed inside the frame $ABCD$, touching the rods BC and CD at the points E and F, respectively.



The system consisting of the frame and the disk is in equilibrium in a vertical plane by two fixed smooth pegs P and Q lying in the same horizontal level, as shown in the figure. It is given that $A\hat{B}C = \frac{2\pi}{3}$, $CE = CF = a$ and the line AOC is vertical.

Show that the magnitude of the reaction exerted on BC by CD at the joint C is $\frac{\sqrt{3}}{2}W$ and find the distance between the two pegs.

(b) The framework shown in the figure consists of five light rods AB, BC, CD, DA and AC that are smoothly jointed at the ends. It is given that $AC = 2a$, $B\hat{A}C = 90^\circ$, $C\hat{D}A = 90^\circ$, $A\hat{B}C = 30^\circ$, and $C\hat{A}D = 30^\circ$. A load W is suspended at the joint B and the framework is smoothly hinged at A to a fixed point and the system is kept in equilibrium in a vertical plane with AC vertical by a force P applied vertically downwards to it at the joint D.



(i) Find the value of P.

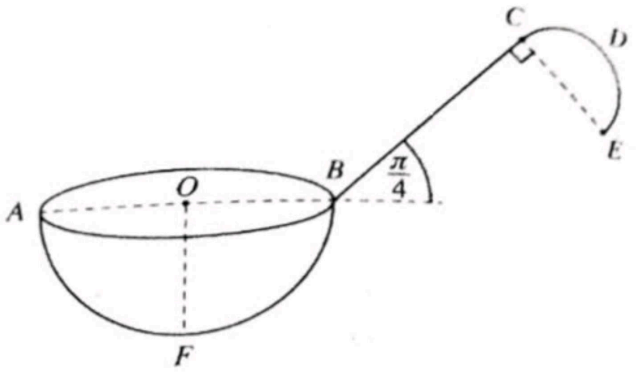
(ii) Draw a stress diagram using Bow's notation for the joints B, C and D. **Hence**, find the stresses in the rods, stating whether they are tensions or thrusts.

16. Show that the centre of mass of

(i) a thin uniform semi-circular wire of radius a is at a distance $\frac{2a}{\pi}$ from its center, and

(ii) a thin uniform hemispherical shell of radius a is at a distance $\frac{a}{2}$ from its center.

A spoon is made by rigidly fixing, a handle made of a thin uniform wire $BCDE$ consisting of a straight section BC of length $2\sqrt{2}a$ and a semi-circular section CDE of radius $\sqrt{2}a$, to a thin uniform hemispherical shell of center O and radius $2a$, as shown in the figure. The diameter CE is perpendicular to BC. The points A and B are ends of a diameter of the circular rim of the hemispherical shell and the point F lies on the surface of the hemispherical shell such that OF is perpendicular to OB. The angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\frac{\pi}{4}$ and the points O, A, B, C, D, E and F lie on the same plane. The mass per unit area of the hemispherical shell is σ and the mass per unit length of the handle is $\sqrt{2}a\sigma$. Show that the center of mass of the spoon lies at a distance $\left(\frac{3\pi-4}{2+5\pi}\right)a$ below OB and a distance $\left(\frac{8+5\pi}{2+5\pi}\right)a$ from OF.



Now, a particle of mass m is fixed to the point A such that the spoon can be kept in equilibrium with OF vertical and the point F touching a horizontal floor. Find m in terms of a and σ.

17.(*a*) A and B are two identical bags. Bag A contains 3 black balls and 2 white balls while bag B contains 4 black balls and 3 white balls. The balls are identical in all aspects except for their colours. Now, two six-sided unbiased dice with faces numbered 1, 2, 3, 4, 5 and 6 are rolled together. If the sum of the numbers then obtained is a prime number, bag A is chosen; if not, bag B is chosen. From the bag chosen, a ball is drawn at random.

(i) Find the probability that the ball drawn is black.

(ii) Given that the ball drawn is black, find the probability that this ball was drawn from bag A.

(*b*) The times taken by 100 students to complete a certain task is summarised in the following table:

Time taken (seconds)	Number of students
0 – 10	10
10 – 20	20
20 – 30	35
30 – 40	20
40 – 50	15

Estimate the median, mean and the variance of the frequency distribution given above.

Later, another 25 students were given the same task. These students fell into each time interval in the above table, 5 students each.

Estimate the mean of the new distribution.

* * *