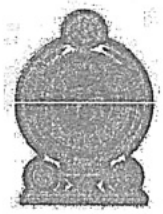
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Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2022 (2023)

10 - Combined Mathematics I

Marking Scheme

https://t.me/mathsworldofNaVi

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{Z}^+$.

For $n=1$, L.H.S. $=\frac{1}{2}$ and R.H.S. $=\frac{1}{2}$. 05

$\therefore$ The result in true for $n=1$. 5

Take any $k \in \mathbb{Z}^+$ and assume that the result is true for $n=k$.

i.e. $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$. ..... (1) 5

Now, $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ 5

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{k+1}{k+2}$. 5

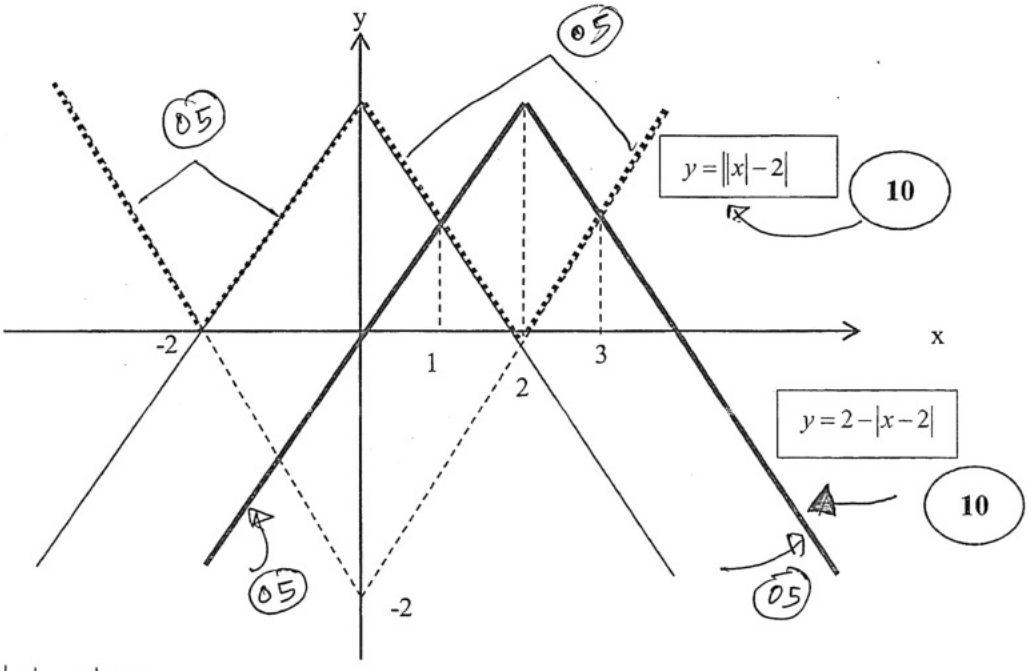
Hence, if the result is true for $n=k$, it is also true for $n=k+1$. We have already proved that the result is true for $n=1$.

Hence, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$. 5

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2. Sketch the graphs of $y=2-|x-2|$ and $y=\big||x|-2\big|$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $\big||x|-2\big|+|x-2|\leq 2$.



$\big||x|-2\big|+|x-2|\leq 2$

$\Leftrightarrow \big||x|-2\big|\leq 2-|x-2|$

From the graph, we have $1\leq x\leq 3$. **5**

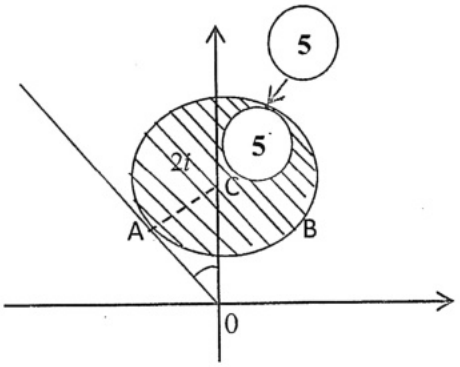
1, 3 find give 05 marks.

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3. Shade in an Argand diagram, the region consisting of points that represent the complex numbers z satisfying the inequality $|\overline{z}+2i| \leq 1$.
Find the greatest value of Arg z for the complex numbers z represented by the points in this shaded region.

$|\overline{z}+2i| = |z-2i|$. → (5)

Hence, the given region is the same as the region given by $|z-2i| \leq 1$.



find the circle — (05)

Let z_0 be the complex number repented by the point A.

From the ΔOAC, we have $A\hat{O}C = \frac{\pi}{6}$. (5)

The required greatest value of Arg z = Arg z_0

$= \frac{\pi}{2} + \frac{\pi}{6}$

$= \frac{2\pi}{3}$. (5)

Direct Give (10) marks'

Aliter for the first (5) :

Let $z = x+iy$, where $x, y \in \mathbb{R}$

Then $|\overline{z}+2i|^2 = |x-(y-z)i|^2$

$= x^2 + (y-2)^2$ (5)

Hence, the given region is the same as the region given by $x^2 + (y-2)^2 \leq 1$.

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4. Let $a \in \mathbb{R}$. Write down the expansion of $(2+ax)^5$ in ascending powers of x up to and including x^2 term. **Hence**, find the values of a for which the coefficient of x^2 in the expansion of $(4-5x)(2+ax)^5$ is -80.

The required expansion $= {}^5C_0 2^5 + {}^5C_1 2^4 (ax) + {}^5C_2 2^3 (ax)^2$ (5)

$= 32 + 5 \times 16ax + 10 \times 8a^2x^2$ (5)

$= 32 + 80ax + 80a^2x^2$

Now, $(4-5x)(2+ax)^5 = 4(2+ax)^5 - 5x(2+ax)^5$

The coefficient of $x^2 = 4 \times 80a^2 - 5 \times 80\,a$ (5)

It is given that $4 \times 80a^2 - 5 \times 80\,a = -80$ (05)

$\therefore\ 4a^2 - 5a + 1 = 0.$

$\therefore\ (4a-1)(a-1) = 0.$

$\therefore\ a = \frac{1}{4}$ or $a = 1.$ (5)

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5. Show that $\lim_{x\to 0}\frac{x((1+x)\operatorname{cosec}2x-\cot 2x)}{\sqrt{1+2x}-\sqrt{1-2x}}=\frac{1}{4}$.

$$\lim_{x\to 0}\frac{x\left((1+x)\operatorname{cosec}2x-\cot 2x\right)}{\left(\sqrt{1+2x}-\sqrt{1-2x}\right)}$$

$$=\lim_{x\to 0}\frac{x}{\sin 2x}\cdot\frac{(1+x-\cos 2x)}{\left(\sqrt{1+2x}-\sqrt{1-2x}\right)}$$ 5

conjugate gives (8)

$$=\lim_{x\to 0}\frac{x}{\sin 2x}\cdot\frac{(1+x-\cos 2x)}{\left(\sqrt{1+2x}-\sqrt{1-2x}\right)}\times\frac{\left(\sqrt{(1+2x)}+\sqrt{1-2x}\right)}{\left(\sqrt{1+2x}+\sqrt{1-2x}\right)}$$ 5

$$=\lim_{x\to 0}\frac{1}{2}\cdot\frac{2x}{\sin 2x}\cdot\frac{\left(2\sin^2 x+x\right)}{\left[(1+2x)-(1-2x)\right]}\cdot\left(\sqrt{1+2x}+\sqrt{1-2x}\right)$$

$\frac{\text{basic}}{\text{rt}}$ ← gives (05)

$$=\lim_{x\to 0}\frac{1}{2}\cdot\frac{2x}{\sin 2x}\cdot\left(\frac{2\sin^2 x}{4x}+\frac{1}{4}\right)\left(\sqrt{1+2x}+\sqrt{1-2x}\right)$$ 5

$$=\frac{1}{2}\times 1\times\frac{1}{4}\times 2$$ 10

$$=\frac{1}{4}.$$

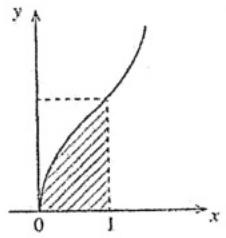
All three limits correct	10
Any two	5

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6. Using $\frac{d}{dx}\left\{x(x^2+1)\tan^{-1}x\right\} = (3x^2+1)\tan^{-1}x + x$, show that $\int_0^1 (3x^2+1)\tan^{-1}x\,dx = \frac{1}{2}(\pi-1)$.

The region enclosed by the curves $y = \sqrt{2(3x^2+1)\tan^{-1}x}$, $x = 1$ and $y = 0$ is rotated about the x-axis through 2π radians. Show that the volume of the solid thus generated is $\pi(\pi-1)$.



Using $\frac{d}{dx}\left\{(x^2+1)\tan^{-1}x\right\} = (3x^2+1)\tan^{-1}x + x$,

we have

$\int_0^1 \left[(3x^2+1)\tan^{-1}x + x\right]dx = x(x^2+1)\tan^{-1}x\Big|_0^1$ **5**

$\therefore \int_0^1 (3x^2+1)\tan^{-1}x\,dx + \int_0^1 x\,dx = 2\tan^{-1}1$

$\therefore \int_0^1 (3x^2+1)\tan^{-1}x\,dx + \frac{x^2}{2}\Big|_0^1 = 2\frac{\pi}{4}$ **5**

$\therefore \int_0^1 (3x^2+1)\tan^{-1}x\,dx = \left(\frac{\pi}{2} - \frac{1}{2}\right)$

$= \frac{1}{2}(\pi-1).$ **5**

No limit gives 05

No limit gives 05

The required volume $= \pi\int_0^1 2\left(3x^2+1\right)\tan^{-1}x\,dx$ **5**

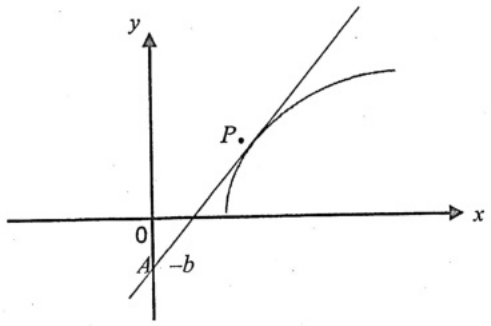
$= 2\pi\frac{1}{2}(\pi-1)$ **5**

$= \pi(\pi-1).$

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7. Let $a, b > 0$. A curve is parametrically given by $x = a\sec\theta$ and $y = b\tan\theta$ for $0 < \theta < \frac{\pi}{2}$. The tangent line to the curve at the point $P \equiv (a\sec\theta, b\tan\theta)$ passes through the point $(0, -b)$. Find the coordinates of P.



$x = a\sec\theta, \qquad y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \quad \frac{dy}{d\theta} = b\sec^2\theta$ (5)

If both correct

$\therefore \frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$ (5)

$\therefore = \frac{b\sec\theta}{a\tan\theta}$. (10)

If direct gives - (10) Marks

The gradient of $AP = \frac{b + b\tan\theta}{a\sec\theta}$.

From the given condition, we have $\frac{b\sec\theta}{a\tan\theta} = \frac{b(1+\tan\theta)}{a\sec\theta}$. (5)

$\therefore \sec^2\theta = \tan\theta + \tan^2\theta$

$\therefore \tan\theta = 1$ (05) (05) given

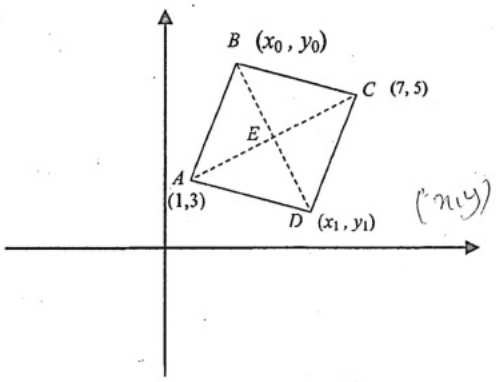
$\therefore \theta = \frac{\pi}{4}$ (5)

$\therefore P \equiv \left(\sqrt{2}a, b\right)$ (5)

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8. Let $ABCD$ be a square with $A \equiv (1,3)$ and $C \equiv (7,5)$. Find the x-coordinates of B and D.



Let $B = (x_0, y_0)$ and $D = (x_1, y_1)$

Since E in the mid-point of AC, we have $E \equiv (4,4)$. **5**

Then, $AE^2 = 3^2+1^2 = 10$

Since $ABCD$ is a square, we have $BE = AE$.

Hence, $(x_0-4)^2+(y_0-4)^2 = 10$. ---------- (1) **5**

Also, $AE \perp BE$.

$\therefore \left(\frac{4-3}{4-1}\right)\times\left(\frac{y_0-4}{x_0-4}\right) = -1.$ **5**

Hence, $y_0 - 4 = -3(x_0-4)$ ---------- (2)

(1) and (2) $\Rightarrow (x_0-4)^2+9(x_0-4)^2 = 10.$ **5**

Hence, $y_0 - 4 = -3(x_0-4)$.

$\therefore\ (x_0-4)^2 = 1.$

$\therefore\ (x_0-4) = \pm 1.$

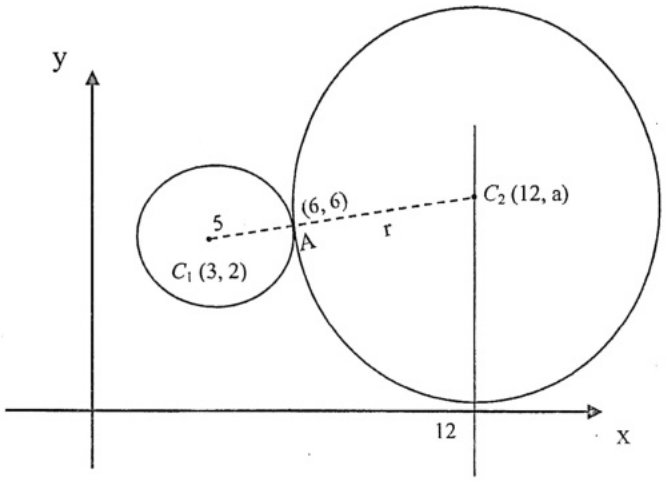
$\therefore\ x_0 = 5$ or $x_0 = 3.$ **5**

Note that (x_1, y_1) also satisfies (1) and (2), when (x_0, y_0) is replaced by (x_1, y_1).

Hence, x coordinates of B and D are 3 and 5.

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9. Find the equation of the circle that touches the circle $x^2+y^2-6x-4y-12=0$ externally at the point (6, 6) and has its centre on the line $x=12$.



Let the center of the given circle be C_1 and the center of the required circle be C_2

Then $C_1 \equiv (3,2)$ and $C_2 \equiv (12,\ a)$; where $a \in \mathbb{R}$ (5)

Since the circles touch externally C_2 lies on the line C_1A.

find Raidus (10)

$\therefore \frac{6-2}{6-3}=\frac{a-6}{12-6}$. (5)

$\therefore 3a-18=24.$

$\therefore a=14.$ (5)

The radius of the required circle $C_2=\sqrt{(12-6)^2+(14-6)^2}$

$=10.$ (5)

Hence, the required equalion is $(x-12)^2+(y-14)^2=100.$ (5)

Me-II

$5+\sqrt{(12-6)^2+(a-6)^2}=\sqrt{(12-3)^2+(a-2)^2}$

$9a^2-252a+1764=0$

$a^2-28a+196=0$

$(a-14)^2=0 \Rightarrow a=14$

14 | 196 | 140

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10. Show that $\cos 5\theta = \cos 3\theta$ if and only if $\theta = \frac{n\pi}{4}$ for $n \in \mathbb{Z}$.

Show also that $\frac{\sin 5\theta - \sin 3\theta}{\cos 5\theta - \cos 3\theta} = -\cot 4\theta$ for $\theta \neq \frac{n\pi}{4}$ and $n \in \mathbb{Z}$.

$\cos 5\theta = \cos 3\theta$

$\Leftrightarrow \quad 5\theta = 2n\pi \pm 3\theta$ for $n \in \mathbb{Z}$. (5) ⟺ not important

$\Leftrightarrow \quad 8\theta = 2n\pi$ or $2\theta = 2n\pi$ for $n \in \mathbb{Z}$.

$\Leftrightarrow \quad \theta = \frac{n\pi}{4}$ or $\theta = n\pi$ for $n \in \mathbb{Z}$. (5)

$\Leftrightarrow \quad \theta = \frac{n\pi}{4}$ for $n \in \mathbb{Z}$.

$$\frac{\sin 5\theta - \sin 3\theta}{\cos 5\theta - \cos 3\theta} = \frac{2\cos 4\theta \sin \theta}{-2\sin 4\theta \sin \theta} \quad (5) \quad (5)$$

$$= -\cot 4\theta \quad (5)$$

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Part B

* Answer **five** questions only.

11. (*a*) Let $0<|p|<1$. Show that the equation $p^2x^2-2x+1=0$ has real distinct roots.
Let α and β $(>\alpha)$ be these roots. Show that α and β are both positive.
Find $(\alpha-1)(\beta-1)$ in terms of p, and **deduce** that $\alpha<1$ and $\beta>1$.
Show that $\sqrt{\beta}-\sqrt{\alpha}=\frac{1}{|p|}\sqrt{2(1-|p|)}$

It is **given that** $\sqrt{\beta}+\sqrt{\alpha}=\frac{1}{|p|}\sqrt{2(1+|p|)}$. Show that the quadratic equation whose roots are $\left|\sqrt{\alpha}-1\right|$ and $\left|\sqrt{\beta}-1\right|$ is $|p|x^2-\sqrt{2(1-|p|)}\,x+\sqrt{2(1+|p|)}-|p|-1=0$.

(*b*) Let $p(x)=2x^3+ax^2+bx-4$, where $a, b\in\mathbb{R}$. It is given that $(x+2)$ is a factor of both $p(x)$ and $p'(x)$, where $p'(x)$ is the derivative of $p(x)$ with respect to x. Find the values of a and b. For these values of a and b, completely factorise $p(x)-3p'(x)$.

(a)

$0<|p|<1$.

Let Δ be the discriminant of $p^2x^2-2x+1=0$.

$\therefore \Delta=4-4p^2=4(1-p^2)>0$, since $p^2<1$. (5) (5)

$\therefore$ The equation has real distinct roots. (5)

≥ 0 Not give 05 marks

$\frac{10}{15}$

15

Let α and β $(>\alpha)$ be these roots.

Then $\alpha\beta=\frac{1}{p^2}>0$. (5)

α and β are both positive or both negative.

But, $\alpha+\beta=\frac{2}{p^2}>0$ gives α and β are both positive. (5) (5)

15

$(\alpha-1)(\beta-1)=\alpha\beta-(\alpha+\beta)+1=\frac{1}{p^2}-\frac{2}{p^2}+1=\frac{p^2-1}{p^2}<0$ and $\alpha-1<\beta-1$.

(5) (5) (5)

since $\beta>\alpha$ gives 05

$\therefore \alpha-1<0$ and $\beta-1>0$. (5)

$\therefore\ \alpha<1$ and $\beta>1$.

20

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$(\sqrt{\beta}-\sqrt{\alpha})^2=\alpha+\beta-2\sqrt{\alpha\beta}=\frac{2}{p^2}-2\frac{1}{|p|}=\frac{2}{p^2}(1-|p|).$ (5) (5)

$\therefore \sqrt{\beta}-\sqrt{\alpha}=\frac{1}{|p|}\sqrt{2(1-|p|)}$ (5)

15

The required equation is $\left(x-\left|\sqrt{\alpha}-1\right|\right)\left(x-\left|\sqrt{\beta}-1\right|\right)=0.$ (10)

$x^2-\left(\left|\sqrt{\alpha}-1\right|+\left|\sqrt{\beta}-1\right|\right)x+\left|\sqrt{\alpha}-1\right|\left|\sqrt{\beta}-1\right|=0$

Since $\left|\sqrt{\alpha}-1\right|=1-\sqrt{\alpha}$ and $\left|\sqrt{\beta}-1\right|=\sqrt{\beta}-1$ we have (05)

$x^2-(\sqrt{\beta}-\sqrt{\alpha})x+\sqrt{\alpha}+\sqrt{\beta}-\sqrt{\alpha\beta}-1=0$ (5)

If subtract given (15) write

If direct given (15)

$\therefore\ x^2-\frac{1}{|p|}\sqrt{2(1-|p|)}\,x+\frac{1}{|p|}\sqrt{2(1+|p|}-\frac{1}{|p|}-1=0$

$\therefore\ |p|x^2-\sqrt{2(1-|p|}\,x+\sqrt{2(1+|p|}-|p|-1=0$ (5)

20

$p(x)=2x^3+ax^2+bx-4$

$\therefore p'(x)=6x^2+2ax+b.$ (5)

Since $(x+2)$ is a factor of $p\ (x)$,

We have $p(-2)=0.$ (5)

Now, $p(-2)=-16+4a-2b-4=0.$ (5)

$\therefore\ 2a-b=10$ ---------- (1)

Since $(x+2)$ is a factor of $p'(x)$,

We have $p'(-2)=0.$ (5)

Now, $p'(-2)=24-4a+b=0.$ (5)

$\therefore 4a-b=24.$ ---------- (2)

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(1) and (2) $\Rightarrow a = 7$ and $b = 4$.

5 5

35

$p(x) - 3p'(x) = (2x^3 + 7x^2 + 4x - 4) - 3(6x^2 + 14x + 4)$ 5

$= (x+2)(2x^2 + 3x - 2) - 3(x+2)(6x+2)$ 5

$= (x+2)\left[2x^2 + 3x - 2 - 18x - 6\right]$

$= (x+2)\ (2x^2 - 15x - 8)$ 5

$= (x+2)(2x+1)(x-8)$

5 5 5

30

Aliter

$p(x) = 2x^3 + ax^2 + bx - 4$

Since $(x+2)$ is a factor of both $p(x)$ and $p'(x)$, we can write

$p(x) = (x+2)^2(2x+k)$. 5 where k is a constant.

10

By comparing the constant terms, we have $4k = -4$

$\therefore k = -1$ 5

$\therefore p(x) = (x+2)^2(2x-1)$.

$\therefore p(x) = (x^2 + 4x + 4)(2x-1) = 2x^3 + 7x^2 + 4x - 4$. 5

By comparing the coefficients of powers of x: $b = 4$ and $a = 7$.

5 5

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$\therefore p(x) = 2x^3 + 7x^2 + 4x - 4$

$\therefore p'(x) = 6x^2 + 14x + 4 = 2(3x^2 + 7x + 2) = 2(x+2)(3x+1)$ **5**

$\therefore p(x) - 3p'(x) = (x+2)^2(2x-1) - 3(2(x+2)(3x+1))$ **5**

$= (x+2)[(x+2)(2x-1) - 6(3x+1)]$

$= (x+2)(2x^2 - 15x - 8)$ **5**

$= (x+2)(2x+1)(x-8)$ **5**

5 **5**

30

Q 11

15	15	20	15	20	35	30

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12. (*a*) Six mangoes and four oranges are to be distributed among eight students so that each student receives at least one fruit.

Find the number of different ways in which

(i) six students get one fruit each and out of the remaining two students one gets **two mangoes** and the other gets **two oranges,**

(ii) seven students get one fruit each, and the other student gets **three mangoes,**

(iii) seven students get one fruit each, and the other student gets **three fruits.**

(*b*) Let $U_r = \dfrac{4(2r+7)}{(2r+1)(2r+3)(2r+5)}$ for $r \in \mathbb{Z}^+$. Also, let $f(r) = \dfrac{A}{(2r+1)} + \dfrac{B}{(2r+3)}$ for $r \in \mathbb{Z}^+$, where A and B are real constants. Determine the values of A and B such that $U_r = f(r) - f(r+1)$ for $r \in \mathbb{Z}^+$.

Hence or otherwise, show that $\sum_{r=1}^{n} U_r = \frac{4}{5} - \frac{3}{2n+3} + \frac{1}{2n+5}$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Hence, find the value of the real constant k such that $\sum_{r=1}^{\infty} (U_r + kU_{r+1}) = 1$.

(*a*) (i)

2 students		6 students	
2M	2OR	4M	2OR
${}^8C_2$	x	${}^6C_4 \times {}^2C_2$	
2OR	2M	4M	2OR
${}^8C_2$	x	${}^6C_4 \times {}^2C_2$	

$C_6^{10} C_1^2 C_1^2 = 840$

25
25

5 5

Answer : $2 \times {}^8C_2 \times {}^6C_4 \times {}^2C_2$ 5

(05) (10) gives

$= 2 \times \dfrac{8!}{6!2!} \times \dfrac{6!}{4!2!} = 2 \times 28 \times 15 = 840$ ways. 5

5

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(ii) එක් සිසුවෙක් | සිසුන් හත්දෙනෙක්

3M | 3M 4OR

$^{8}C_{1}$ x $^{7}C_{3} \times {}^{4}C_{4}$

5

පිළිතුර: $^{8}C_{1} \times {}^{7}C_{3} \times {}^{4}C_{4} = 8 \times 4\frac{7!}{4!3!} = 8 \times 35 = 280$

5 5

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(iii) පළතුරු 3ක්: 3M + 3OR | 2M + 1OR | 1M + 2OR 5

අවස්ථා 4 ක්

3M (ii) හි පරිදි විධි 280 යි

3OR $^{8}c_{1} \times {}^{7}c_{6} \times {}^{1}c_{1} = 8 \times 7 = 56$ 5

2M + 1OR $^{8}c_{1} \times {}^{7}c_{4} \times {}^{3}c_{3} = 8 \times 35 = 280$ 5

1M + 2OR $^{8}c_{1} \times {}^{7}c_{5} \times {}^{2}c_{2} = 8 \times 21 = 168$ 5

නැ ඇණිනෙය වෙන්නෙ වීය අවස්ථා ගණන - 05

පිළිතුර $= 280 + 56 + 280 + 168$

$= 784$ 5

25

Enu

වෙනත් ක්‍රමයක්

(a) අඹ 6 යි. දොඩම් 4යි. සිසුන් 8යි.

(i) එක් සිසුවෙකුට අඹ දෙකකුත් තවත් සිසුවෙකුට දොඩම් දෙකකුත් දෙන නිසා ඉතිරි සිසුන් 6 දෙනාට අඹ හතරකුත් දෙඩම් දෙකකුත් ඉතිරිව ඇත.

						2Ma	2Or

සිසුන් 6

සිසුන් 6 දෙනෙකු අතර අඹ 4ක් හා දොඩම් 2ක්, පළතුරු එක බැගින් බෙදා දිය හැකි ක්‍රම ගණන

$= \frac{6!}{4!2!}$ 10

$^{6}C_{2}$ → 05

$\frac{6!}{4!2!}$ → 05

සිසුන් 8 දෙනෙකු අතරින් එක් සිසුවෙකු තෝරා අඹ 2ක් දිය හැකි විධි ගණන $= {}^8C_1$

(5) සිසුන් 7 දෙනෙකු අතරින් එක් සිසුවෙකු තෝරා දොඩම් 2ක් දිය හැකි විධි ගණන $= {}^7C_1$

$$\text{පිළිතුර} = \frac{6!}{4!2!} \times {}^8C_1 \times {}^7C_1 \quad (5)$$

$$= 840 \quad (5)$$

හෝ

$$= \frac{6!}{4!2!} \times {}^8P_2$$

$$= 840$$

25

(ii) එක් සිසුවෙකුට අඹ 3කුත් අනෙක් සිසුන් 7 දෙනාට එක පළතුර බැගින්:

							3Ma

සිසුන් 7 දෙනෙකු අතර අඹ 3ක් හා දොඩම් 4ක්, පළතුරු එක බැගින් බෙදා දිය හැකි ක්‍රම ගණන

$$= \frac{7!}{4!3!}$$

සිසුන් 8 දෙනෙකු අතරින් එක් සිසුවෙකු තෝරා අඹ 3ක් දිය හැකි විධි ගණන $= {}^8C_1$

$$\therefore \text{පිළිතුර} = {}^8C_1 \times \frac{7!}{4!3!}$$

$$= 280$$

(iii)

එක් සිසුවෙකුට පළතුරු 3		සිසුන් 7 දෙනාට පළතුරු 7		විධි ගණන
අඹ	දොඩම්	අඹ	දොඩම්	
3	0	3	4	$= {}^8C_1 \times \frac{7!}{3!4!} = 280$ (5)
2	1	4	3	$= {}^8C_1 \times \frac{7!}{4!3!} = 280$ (5)
1	2	5	2	$= {}^8C_1 \times \frac{7!}{5!2!} = 168$ (5)
0	3	6	1	$= {}^8C_1 \times \frac{7!}{6!} = 56$ (5)

මුළු විධි ගණන

$$= 280 + 280 + 168 + 56$$

$$= 784 \quad (5)$$

25

Total Number of ways

$= 280 + 280 + 168 + 56$

$= 784$ (5)

25

(*b*). $r \in \mathbb{Z}^+$

$$U_r = \frac{4(2r+7)}{(2r+1)(2r+3)(2r+5)}$$

$$U_r = f(r) - f(r+1)$$

$$\frac{4(2r+7)}{(2r+1)(2r+3)(2r+5)} = \frac{A}{2r+1} + \frac{B}{2r+3} - \frac{A}{2r+3} - \frac{B}{2r+5}$$ (5)

$$\therefore 4(2r+7) = A(2r+3)(2r+5) + (B-A)(2r+1)(2r+5) - B(2r+1)(2r+3)$$

$$= (4A+4B)r + 10A - 2B$$

Any Method (10)

Comparing coefficient of powers of r:

r : $8 = 4A + 4B \Rightarrow 2 = A + B$ (5)

r^0 : $28 = 10A + 2B \Rightarrow 14 = 5A + B$ (5)

$A = 3, \quad B = -1$

25

$U_r = f(r) - f(r+1)$ where $f(r) = \frac{3}{2r+1} - \frac{1}{2r+3}$ (5)

$r = 1; \quad U_1 = f(1) - f(2)$

$r = 2; \quad U_2 = f(2) - f(3)$ (5)

. .

. .

. .

$r = n-1; \quad U_{n-1} = f(n-1) - f(n)$

$r = n; \quad U_n = f(n) - f(n+1)$ (5)

$$\sum_{r=1}^{n} U_r = f(1) - f(n+1)$$ (5)

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$$\therefore \quad \sum_{r=1}^{n} U_r = f(1) - f(n+1)$$

(5)

$$= 1 - \frac{1}{5} - \frac{3}{2n+3} + \frac{1}{2n+5}$$

$$= \frac{4}{5} - \frac{3}{2n+3} + \frac{1}{2n+5} \quad (5) \quad r \in \mathbb{Z}$$

30

Important

$$\lim_{n \to \infty} \sum_{r=1}^{n} U_r \quad (5)$$

$$= \lim_{n \to \infty} \left(\frac{4}{5} - \frac{3}{2n+3} + \frac{1}{2n+5} \right)$$

$$= \frac{4}{5} \quad (5) \quad (+5)$$

$\therefore$ The infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and the sum is $\frac{4}{5}$. (5)

15

$$1 = \sum_{r=1}^{\alpha} (U_r + kU_{r+1})$$

$$= (1+k)\left(\sum_{r=1}^{\alpha} U_r \right) - kU_1 \quad (5)$$

$$= (1+k)\left(\frac{4}{5} \right) - k\left(\frac{12}{35} \right) \quad (5)$$

gives (10) marks

$$\therefore k = \frac{7}{16} \quad (5)$$

15

A, B wrong gives (20) Marks

Q12	25	15	25	25	30	15	15

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13. (*a*) Let $\mathbf{A} = \begin{pmatrix} a & -2 \\ 1 & a+2 \end{pmatrix}$. Show that $\mathbf{A}^{-1}$ exists for all $a \in \mathbb{R}$.

The matrices $\mathbf{P} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 7 & 4 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ are such that $\mathbf{A} = \mathbf{PQ}^{\mathbf{T}} + \mathbf{R}$. Show that $a = 1$.

For this value of a, write down $\mathbf{A}^{-1}$ and **hence**, find the values of x and y such that $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$.

(*b*) Let $z, w \in \mathbb{C}$. Show that $z\bar{z} = |z|^2$ and **hence**, show that $|z+w|^2 = |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2$.

Deduce that $|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$ and give a geometric interpretation for it when the points representing z, w and 0 in the Argand diagram are non-collinear.

(*c*) Let $z = -1 + \sqrt{3}i$. Express z in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $\frac{\pi}{2} < \theta < \pi$.

Let $z^n = a_n + ib_n$, where $a_n, b_n \in \mathbb{R}$ for $n \in \mathbb{Z}^+$. Write down $\mathrm{Re}\left(z^m \cdot z^n\right)$ in terms of a_m, a_n, b_m and b_n for $m, n \in \mathbb{Z}^+$.

Considering z^{m+n} and using De Moivre's theorem, show that $a_m a_n - b_m b_n = 2^{m+n}\cos(m+n)\frac{2\pi}{3}$, for $m, n \in \mathbb{Z}^+$.

(*a*) $|A| = a(a+2) + 2 = a^2 + 2a + 2 = (a+1)^2 + 1 \neq 0$ for all $a \in \mathbb{R}$., (5)

(5) gives −15

$\therefore A^{-1}$ exists for all $a \in \mathbb{R}$ (5)

15

$A = PQ^T + R$ (5)

Q^T − (5)

Substitute − (05)

$\begin{pmatrix} \alpha & -2 \\ 1 & a+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 7 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ (5)

(5)

$= \begin{pmatrix} 0 & -5 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

(05)

$= \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ (5)

$a = 1$ and $a + 2 = 3$. $\therefore a = 1$ (5)

25

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When $a=1$, $A=\begin{pmatrix}1 & -2\\ 1 & 3\end{pmatrix}$ $\therefore A^{-1}=\frac{1}{5}\begin{pmatrix}3 & 2\\ -1 & 1\end{pmatrix}$ 10

$$A\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-5\\ 10\end{pmatrix}$$

$$\therefore \quad \begin{pmatrix}x\\ y\end{pmatrix}=A^{-1}\begin{pmatrix}-5\\ 10\end{pmatrix}=\frac{1}{5}\begin{pmatrix}3 & 2\\ -1 & 1\end{pmatrix}\begin{pmatrix}-5\\ 10\end{pmatrix}=\begin{pmatrix}1\\ 3\end{pmatrix}$$ 10

$\therefore$ 5 $x=1$ and 5 $y=3$. **30**

(*b*) Taking $z=x+iy;\ x,y\in\mathbb{R}$, we have

$z\bar{z}=(x+iy)(x-iy)=x^2-i^2y^2=x^2+y^2=|z|^2$

5 5 **10**

$|z+w|^2=(z+w)\overline{(z+w)}$ 5

$=(z+w)(\bar{z}+\bar{w})$ 5

$=z\bar{z}+z\bar{w}+\bar{z}w+w\bar{w}$ 5

$=|z|^2+z\bar{w}+\overline{z\bar{w}}+|w|^2$ 5

$=|z|^2+2\,\mathrm{Re}(z\bar{w})+|w|^2$ ---------- (i) **20**

(i) with w replace by -w, 5

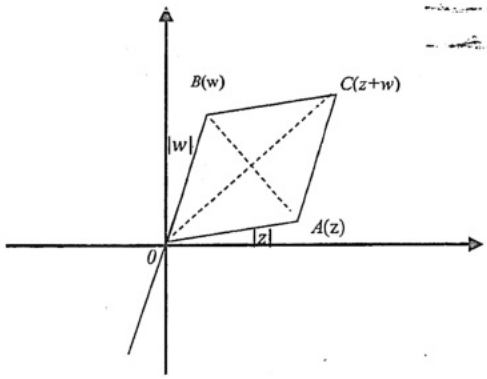
$|z-w|^2=|z|^2-2\,\mathrm{Re}(z\bar{w})+|w|^2$ ---------- (ii)

$\therefore$ By (i) and (ii)

$|z+w|^2+|z-w|^2=2\left(|z|^2+|w|^2\right)$. 5

15	25	30	10	20	15	15	05	15

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If z, w and 0 are non-collinear, then $OC^2 + AB^2 = 2\left(OA^2 + OB^2\right)$.

$\left(\because OC = |z+w| \text{ and } AB = |z+w|.\right)$

In a parallelogram, sum of the squares of the diagonals is equal to twice the sum of the squares of its sides. 5

15

(c) $z = -1 + \sqrt{3}i = 2\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ 5 10

Here $r = 2$, and $\theta = \frac{2\pi}{3}$.

15

$\text{Re}(z^m z^n) = \text{Re}\left[\left(a_m + ib_m\right)\left(a_n + ib_n\right)\right] = a_m a_n - b_m b_n$ 5 -------- (1)

05

$z^m z^n = z^{m+n} = \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{n+m} = 2^{m+n}\left[\cos\frac{2(m+n)\pi}{3} + i\sin\frac{2(m+n)\pi}{3}\right]$ 5 5

$\therefore \text{Re}(z^m z^n) = 2^{m+n}\cos(m+n)\frac{2\pi}{3}$ ---------- (2) 5

(1) and (2) $\Rightarrow a_m a_n - b_m b_n = 2^{m+n}\cos(m+n)\frac{2\pi}{3}$.

15

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14.(*a*) Let $f(x)=\frac{2x+3}{(x+2)^2}$ for $x \neq -2$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\frac{-2(x+1)}{(x+2)^3}$ for $x \neq -2$.

Hence, find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Also, find the coordinates of the turning point of $f(x)$.

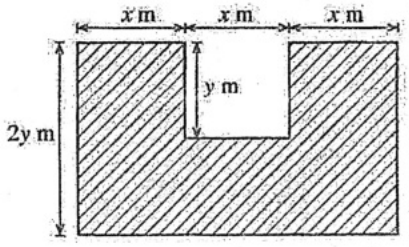
It is given that $f''(x)=\frac{2(2x+1)}{(x+2)^4}$ for $x \neq -2$. Find the coordinates of the point of inflection of the graph of $y=f(x)$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, the turning point and the point of inflection.

State the smallest value of k for which $f(x)$ is one-one on $[k, \infty)$.

(*b*) The shaded region shown in the figure is of area 45 m^2. It is obtained by removing a rectangle of length x m and width y m from a rectangle of length $3x$ m and width $2y$ m. Show that the perimeter L m of the shaded region is given by $L=6x+\frac{54}{x}$ for $x>0$.

Find the value of x such that L is minimum.



(a) For $x \neq -2$, $f(x)=\frac{2x+3}{(x+2)^2}$.

$$f'(x)=\frac{(x+2)^2(2)-2(2x+3)(x+2)}{(x+2)^4}$$ (05) (05) (05) 20

$$=\frac{2(x+2)[x+2-2x-3]}{(x+2)^4}$$ x + 2 - 2x - 3 (05)

$$=\frac{-2(x+1)}{(x+2)^3}$$ 5

25

$f'(x)=0 \Leftrightarrow x=-1$ 5

	$-\infty<x<-2$	$-2<x<-1$	$-1<x<\infty$
sign of $f'(x)$	(−)	(+)	(−)
$f(x)$ is	decreasing ↘	increasing ↗	decreasing ↘
	5	5	5

$\therefore f(x)$ is increasing on $(-2,-1]$ and decreasing on $(-\infty,-2)$ and $[-1,\infty)$.

20

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Turning point: $(-1,1)$ is a local maximum.

5 5

$$f''(x)=\frac{2(2x+1)}{(x+2)^4}$$

$$f''(x)=0 \Leftrightarrow x=\frac{-1}{2}.$$ 5

	$-2<x<-\frac{1}{2}$	$-\frac{1}{2}<x<\infty$
Sign of $f''(x)$	(−)	(+)
Concavity	concave down	concave up

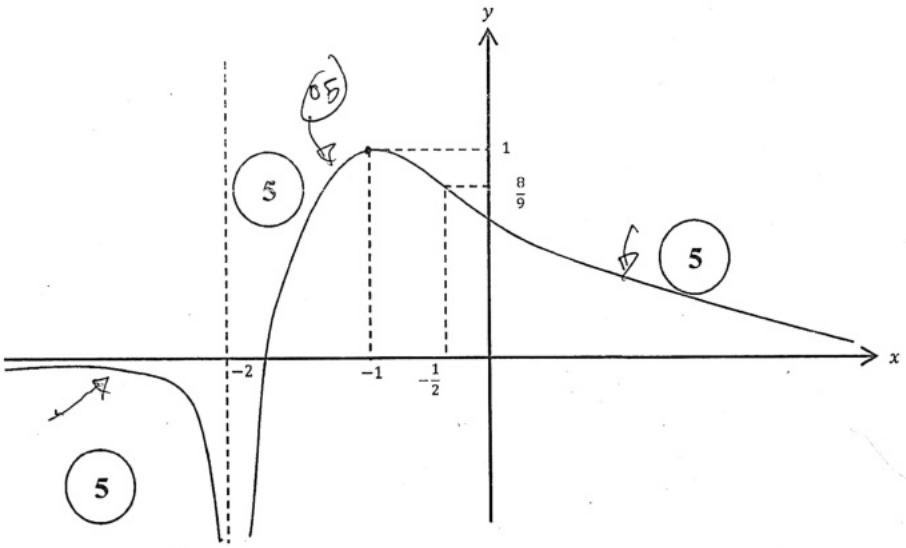
5 5

$\therefore$ the point of inflection is $\left(\frac{-1}{2},\frac{8}{9}\right)$ 5

Horizontal Asymptote: $\lim_{x\to\pm\infty} f(x)=0$ $\therefore y=0$ 5

Vertical Asymptote : $x=-2$ 5

$\left(\lim_{x\to-2^+} f(x)=-\infty \text{ and } \lim_{x\to-2^-} f(x)=-\infty\right)$



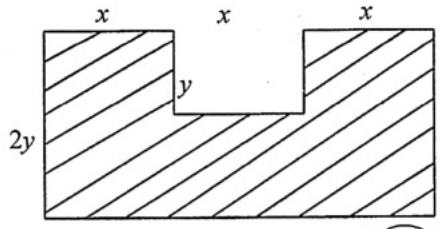
55

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Smallest value of k, , for which $f(x)$ is one to one on $[k,\infty)$ is $k=-1$. (5)

05

(*b*)



for $x>0, \quad y>0$

(5) (5)

Area of the shaded region, $45=(3x)(2y)-xy$

$\therefore \quad 45=5xy$

$\therefore \quad y=\frac{9}{x}$ (5)

$L=6x+6y$ (10) / 0

$=6x+\frac{54}{x}$ for $x>0$ (5)

$\frac{dL}{dx}=6-\frac{54}{x^2}=\frac{6(x^2-9)}{x^2}=\frac{6(x-3)(x+3)}{x^2}$

(5)

$\frac{dL}{dx}=0 \quad \Leftrightarrow \quad x=3$ (5)

For $0<x<3, \quad \frac{dL}{dx}<0$ and

For $x>3, \quad \frac{dL}{dx}>0.$ (5)

$\therefore$ L is minimum when $x=3$.

45

Q14 | 25 | 20 | 55 | 05 | 45 |

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15. (*a*) Find the values of the constants A, B and C such that

$x^2+x+2=A(x^2+x+1)+(Bx+C)(x+1)$ for all $x\in\mathbb{R}$.

Hence, write down $\dfrac{x^2+x+2}{(x^2+x+1)(x+1)}$ in partial fractions and find $\displaystyle\int\frac{x^2+x+2}{(x^2+x+1)(x+1)}\,dx$.

(*b*) Show that $1+\sin 2x=2\cos^2\left(\frac{\pi}{4}-x\right)$ and **hence,** show that $\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin 2x}dx=1$.

(*c*) Let $I=\displaystyle\int_0^{\frac{\pi}{2}}\frac{x^2\cos 2x}{(1+\sin 2x)^2}\,dx$. Using integration by parts, show that $I=-\frac{\pi^2}{8}+J$, where $J=\displaystyle\int_0^{\frac{\pi}{2}}\frac{x}{1+\sin 2x}dx$.

Using the relation $\displaystyle\int_0^a f(x)dx=\int_0^a f(a-x)dx$ and the result in (*b*), evaluate J and show that $I=\frac{\pi}{8}(2-\pi)$.

(*a*)

$$x^2+x+2=A(x^2+x+1)+(Bx+C)(x+1)$$
$$=(A+B)x^2+(A+B+C)x+A+C$$

Comparing the coefficients of powers of x:

x^0 : $2=A+C$

x : $1=A+B+C$ } 5

x^2 : $1=A+B$

$\therefore A=2,\quad B=-1 \quad and \quad C=0.$ (5)

(5) (5)

20

$$\frac{x^2+x+2}{(x^2+x+1)(x+1)}=\frac{2}{x+1}-\frac{x}{x^2+x+1}$$ (5)

$$\therefore\int\frac{x^2+x+2}{(x^2+x+1)(x+1)}dx=2\int\frac{1}{x+1}dx-\int\frac{x}{x^2+x+1}dx$$ (5)

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$$= 2\ln|x+1| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

modulus important (5) (5) (5)

$$-\frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{(x+\frac{1}{2})}{\frac{\sqrt{3}}{2}} + C \quad (5)$$

(5) (5)

$x^2+x+1>0$

$$= 2\ln|x+1| - \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{(2x+1)}{\sqrt{3}} + C, \text{ where } C \text{ is an arbitrary constant.}$$

40

(*b*)

$$2\cos^2\left(\frac{\pi}{4}-x\right) = 2\left(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\right)^2$$

$$= (\cos x + \sin x)^2 \quad (5)$$

$$= 1 + 2\sin x\cos x \quad (5)$$

$$= 1 + \sin 2x \quad (5)$$

15

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin 2x}dx = \int_0^{\frac{\pi}{2}} \frac{1}{2\cos^2\left(\frac{\pi}{4}-x\right)}dx \quad (5)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{\pi}{4}-x\right)dx \quad (5)$$

$$= \frac{-1}{2}\tan\left(\frac{\pi}{4}-x\right)\Big|_0^{\frac{\pi}{2}} \quad (5)$$

$$= \frac{-1}{2}\left(\tan\left(\frac{-\pi}{4}\right) - \tan\frac{\pi}{4}\right) \quad (5)$$

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$$= \frac{-1}{2}(-1-1)$$

$$= 1 \quad (5)$$

25

(C) $$I = \int_0^{\frac{\pi}{2}} \frac{x^2 \cos 2x}{(1+\sin 2x)^2} dx$$

$$= x^2\left(\frac{-1}{2}\right)\frac{1}{1+\sin 2x}\Bigg|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x}{1+\sin 2x} dx \quad (5)$$

(5) (5)

$$= \frac{-1}{2} \times \frac{\pi^2}{4} \times \frac{1}{1+0} + \int_0^{\frac{\pi}{2}} \frac{x}{1+\sin 2x} dx$$

(5)

$$= \frac{-\pi^2}{8} + \int_0^{\frac{\pi}{2}} \frac{x}{1+\sin 2x} dx$$

$$= \frac{-\pi^2}{8} + J. \quad (5)$$

25

$$J = \int_0^{\frac{\pi}{2}} \frac{x}{1+\sin 2x} dx = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\underbrace{1+\sin 2\left(\frac{\pi}{2}-x\right)}_{\sin 2x}} dx \quad (5)$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin 2x} dx - \int_0^{\frac{\pi}{2}} \frac{x}{\underbrace{1+\sin 2x}_{J}} dx \quad (5)$$

$$\therefore 2J = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\underbrace{1+\sin 2x}_{1}} dx \quad (5)$$

$$\therefore J = \frac{\pi}{4} \quad (5)$$

$$\therefore I = \frac{-\pi^2}{8} + \frac{\pi}{4} = \frac{\pi}{8}(2-\pi) \quad (5)$$

25

Q15 | 20 | 40 | 15 | 25 | 25 | 25 |

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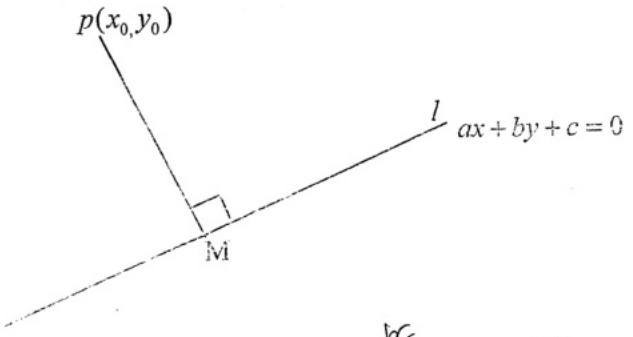
16. Let $P \equiv (x_0, y_0)$ and l be the straight line given by $ax+by+c=0$. Show that the perpendicular distance from P to l is $\dfrac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$.

Let l_1 and l_2 be two straight lines given by $4x-3y+8=0$ and $3x-4y+13=0$, respectively. Show that l_1 and l_2 intersect at $A \equiv (1, 4)$.

Also, show that the parametic equations of the bisector of the acute angle between l_1 and l_2 can be written as $x=t$ and $y=t+3$, where $t \in \mathbb{R}$.

Hence, show that the equation of any circle touching both straight lines l_1 and l_2, and lying in the region between l_1 and l_2 that contains the acute angle, is given by $(x-t)^2+(y-t-3)^2=\frac{1}{25}(t-1)^2$, where $t \in \mathbb{R}$ and $t \neq 1$.

From among the above circles, find the equations of the circles that intersect the circle centred at A of radius 1, orthogonally.



Here $a^2+b^2 \neq 0$

Equation of the line PM is $(y-y_0)=\left(\frac{a}{b}\right)(x-x_0)$ (5)

Any point on the line passing through P and perpendicular to l can be written as

(x_0+at, y_0+bt) for $t \in \mathbb{R}$. (5)

M is on l; $a(x_0+at)+b(y_0+bt)+c=0$ (5)

$$\therefore t(a^2+b^2)=-ax_0+by_0+c$$

$$\therefore \quad t=\frac{-(ax_0+by_0+c)}{a^2+b^2}$$ (5)

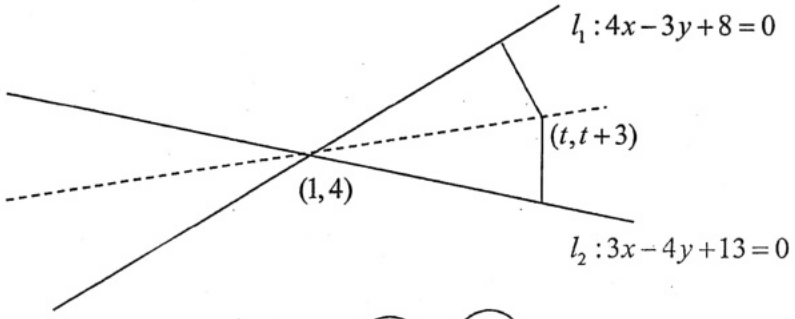
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$\therefore$ Required distance $PM = \sqrt{a^2t^2 + b^2t^2}$ (5)

$= \sqrt{a^2 + b^2}\,|t|$

$= \dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$ (5)

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(5) (5)

Substituting the coordinates of A, to l_1, and l_2 we get l_1 and l_2 intersect at $A = (1,4)$ (5)

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The angle bisectors are given by $\dfrac{4x-3y+8}{5} = \pm\dfrac{3x-4y+13}{5}$ (10)

The angle bisectors are $\underbrace{x+y-5=0}_{m=-1}$ and $x-y+3=0$.

(5) (5)

Let θ be the acute angle between l_1 and $x_1 + y - 5 = 0$

Then, $\tan\theta = \left|\dfrac{\frac{4}{3}-(-1)}{1+\frac{4}{3}(-1)}\right| = 7 > 1$ (5)

(10) (5)

Not modulus give (5)

$\therefore$ The acute angle bisector is $x - y + 3 = 0$ (5)

The acute angle bisector parametrically is given below.

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Let $x = t$ for $t \in \mathbb{R}$. (5)

Then $y = x + 3 = t + 3$. (5)

55

Centre of the required any circle must be on the acute angle bisector.

(5)

$\therefore$ Centre is of the form $(t, t+3)$ for $t \in \mathbb{R}$

modulus important

$$\text{Radius} = \frac{|4t - 3(t+3) + 8|}{5} = \left|\frac{t-1}{5}\right|$$

(5) (5)

$\therefore$ The equation is

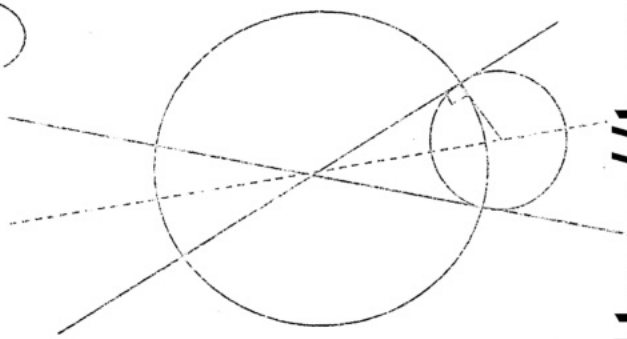
$$(x-t)^2 + (y-(t+3))^2 = \tfrac{1}{25}(t-1)^2 \quad (5)$$

That is $(x-t)^2 + (y-t-3)^2 = \frac{1}{25}(t-1)^2$, where $t \in \mathbb{R}$.

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Applying Pythagoras theorem to orthogonally intersecting circles gives

$$(t-1)^2 + (t+3-4)^2 = 1^2 + \frac{1}{25}(t-1)^2 \quad (10)$$



$$\therefore \quad (t-1)^2 = \frac{25}{49}$$

$$\Rightarrow t - 1 = \frac{5}{7} \quad \text{or} \quad t - 1 = \frac{-5}{7}$$

(5) (5)

$$\therefore \quad t = \frac{12}{7} \quad \text{or} \quad t = \frac{2}{7}$$

$\therefore$ Equations of circles that intersects S orthogonally are

$$\left(x - \frac{12}{7}\right)^2 + \left(y - \frac{33}{7}\right)^2 = \frac{1}{25}\left(\frac{12}{7} - 1\right)^2 \qquad \left(\text{when } t = \frac{12}{7}\right)$$

(5)

$$(7x-12)^2 + (7y-33)^2 = 1$$

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$\left(x-\frac{2}{7}\right)^2+\left(y-\frac{23}{7}\right)^2=\frac{1}{25}\left(\frac{2}{7}-1\right)^2 \qquad \left(\text{when } t=\frac{2}{7}\right)$

$(7x-2)^2+(7y-23)^2=1$ (5)

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Q.16

30	15	55	20	30

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17. (*a*) Write down $\cos(A+B)$ in terms of $\cos A$, $\cos B$, $\sin A$ and $\sin B$, and obtain a similar expression for $\sin(A-B)$.

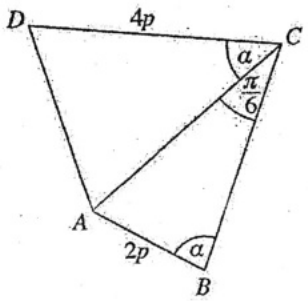
Let $k \in \mathbb{R}$ and $k \neq 1$. By separately considering the cases $k > 1$ and $k < 1$, express

$2k\cos\left(\theta+\frac{\pi}{3}\right)+2\sin\left(\theta-\frac{\pi}{6}\right)$ in the form $R\cos(\theta+\alpha)$, where $R(>0)$ in terms of k, and $\alpha(0<\alpha<2\pi)$ are real constants to be determined.

Hence, solve $2k\cos\left(\theta+\frac{\pi}{3}\right)+2\sin\left(\theta-\frac{\pi}{6}\right)=|k-1|$.

(*b*) In the quadrilateral $ABCD$ shown in the figure $AB=2p$, $CD=4p$, $A\hat{C}B=\frac{\pi}{6}$ and $A\hat{B}C=A\hat{C}D=\alpha$. Show that $AD^2=16p^2(\sin^2\alpha-\sin 2\alpha+1)$.

Hence, show that if $AD=4p$, then $\alpha=\tan^{-1}(2)$.



(*c*) Solve, $\tan^{-1}(\ln x^{\frac{2}{3}})+\tan^{-1}(\ln x)+\tan^{-1}(\ln x^2)=\frac{\pi}{2}$ for $x>1$.

direct working gmc (5)

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

(*a*) $\cos(A+B)=\cos A\cos B-\sin A\sin B$ (5)

$\sin(A-B)=\cos\left(\frac{\pi}{2}-(A-B)\right)$ (5)

$=\cos\left((\frac{\pi}{2}-A)+B\right)$

$=\cos\left(\frac{\pi}{2}-A\right)\cos B-\sin\left(\frac{\pi}{2}-A\right)\sin B$ (5)

$=\sin A\cos B-\cos A\sin B$ (5)

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$2k\cos\left(\theta+\frac{\pi}{3}\right)+2\sin\left(\theta-\frac{\pi}{6}\right)$ (5)

$=2k\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)+2\left(\sin\theta\cos\frac{\pi}{6}-\cos\theta\sin\frac{\pi}{6}\right)$ (5)

$=k\left(\cos\theta-\sqrt{3}\sin\theta\right)+\left(\sqrt{3}\sin\theta-\cos\theta\right)$ (5)

$=(k-1)\left(\cos\theta-\sqrt{3}\sin\theta\right)$

$=2(k-1)\left(\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)$ (5)

$=2(k-1)\cos(\theta+\beta)$ where $\beta=\frac{\pi}{3}$ (5)

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when $k>1$ $\quad 2k\cos\left(\theta+\frac{\pi}{3}\right)+2\sin\left(\theta-\frac{\pi}{6}\right)=2(k-1)\cos\left(\theta+\frac{\pi}{3}\right)$

where $R=2(k-1)$ and $\alpha=\frac{\pi}{3}$. (5)

when $k<1$ $\quad 2k\cos\left(\theta+\frac{\pi}{3}\right)+2\sin\left(\theta-\frac{\pi}{6}\right)=2(1-k)\cos\left(\pi+\theta+\frac{\pi}{3}\right)$

$=2(1-k)\cos\left(\theta+\frac{4\pi}{3}\right)$

where $R=2(1-k)$ and $\alpha=\frac{4\pi}{3}$. (5)

35

$2k\cos\left(\theta+\frac{\pi}{3}\right)+2\sin\left(\theta-\frac{\pi}{6}\right)=|k-1|$

when $k>1$

$2(k-1)\cos\left(\theta+\frac{\pi}{3}\right)=k-1$

$\therefore \cos\left(\theta+\frac{\pi}{3}\right)=\frac{1}{2}$ (5)

$\Rightarrow \theta+\frac{\pi}{3}=2n\pi\pm\frac{\pi}{3} \qquad n\in\mathbb{Z}$

$\therefore \quad \theta=2n\pi-\frac{\pi}{3}\pm\frac{\pi}{3}$, where $n\in\mathbb{Z}$. (5)

when $k<1$

$2(1-k)\cos\left(\theta+\frac{4\pi}{3}\right)=1-k$

$\therefore \quad \cos\left(\theta+\frac{4\pi}{3}\right)=\frac{1}{2}$ (5)

$\theta+\frac{4\pi}{3}=2n\pi\pm\frac{\pi}{3} n\in\mathbb{Z}.$

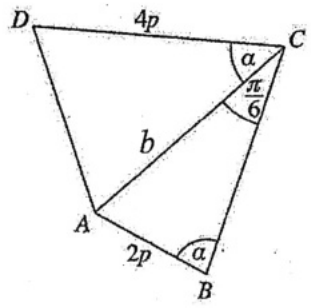
$\therefore \quad \theta=2n\pi-\frac{4\pi}{3}\pm\frac{\pi}{3}$, where $n\in\mathbb{Z}$. (5)

20

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(*b*) Sine Rule for the triangle ABC: **10**



$$\frac{b}{\sin\alpha}=\frac{2p}{\sin\frac{\pi}{6}} \Rightarrow b=4p\sin\alpha$$ **5**

Cosine Rule for the triangle ACD: **10**

$$AD^2=b^2+(4p)^2-2b(4p)\cos\alpha$$
$$=16p^2\sin^2\alpha+16p^2-2(4p)^2\sin\alpha\cos\alpha$$
$$=16p^2(\sin^2\alpha-\sin 2\alpha+1)$$ **5**

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If $AD=4p$, the ADC is an isosceles triangle, we have

$$\sin^2\alpha-\sin 2\alpha+1=1$$
$$\sin\alpha(\sin\alpha-2\cos\alpha)=0$$ **5**

Since $\sin\alpha\neq 0$,

$$\sin\alpha=2\cos\alpha$$ **5**

$$\frac{\sin\alpha}{\cos\alpha}=2 \qquad \cos\alpha\neq 0$$

$\therefore$ $\tan\alpha=2$

$\alpha=\tan^{-1}(2)$ **5**

15

Q17 | 20 | 35 | 20 | 30 | 15 | 30

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(c)

For $x>1$

$$\underbrace{\tan^{-1}\left(\ln x^{\frac{2}{3}}\right)}_{\alpha}+\underbrace{\tan^{-1}\left(\ln x\right)}_{\beta}+\underbrace{\tan^{-1}\left(\ln x^{2}\right)}_{\theta}=\frac{\pi}{2}$$

$$\beta+\theta=\frac{\pi}{2}-\alpha \quad \textbf{5}$$

$$\tan(\beta+\theta)=\cot\alpha \quad \textbf{5}$$

$$\frac{\tan\beta+\tan\theta}{1-\tan\beta\tan\theta}=\frac{1}{\tan\alpha} \quad \textbf{5}$$

$$\therefore \quad \frac{\ln x+\ln x^{2}}{1-\ln x\ln x^{2}}=\frac{1}{\ln x^{\frac{2}{3}}} \quad \textbf{5}$$

$$\frac{\ln x^{3}}{1-2(\ln x)^{2}}=\frac{1}{\frac{2}{3}\ln x}$$

Taking $t=\ln x$,

$$3\times\frac{2}{3}t^{2}=1-2t^{2} \quad \textbf{5}$$

$$4t^{2}=1$$

$$\ln x=t=\frac{1}{2} \qquad \left(\therefore t\neq\frac{-1}{2} \text{ as } t=\ln x \text{ and } x>1\right)$$

$$\therefore \quad x=e^{\frac{1}{2}} \quad \textbf{5}$$

30

Verification

$$\tan^{-1}\left(\ln\left(e^{\frac{1}{2}}\right)^{\frac{2}{3}}\right)+\tan^{-1}\left(\ln e^{\frac{1}{2}}\right)+\tan^{-1}\left(\ln e\right)\doteq\frac{\pi}{2}.$$

$$\Leftrightarrow \qquad \underbrace{\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{2}\right)}_{\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}}}\doteq\frac{\pi}{4}.$$

$$=\frac{\frac{5}{6}}{\frac{5}{6}}$$

$$=1$$

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