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Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2021 (2022)

10 - Combined Mathematics I

Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

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G. C. E (Advanced Level) Examination - 2021(2022)

10 - Combined Mathematics I

Distribution of Marks

Paper I

Part A	$= 10 \times 25$	$=$	**250**
Part B	$= 05 \times 150$	$=$	**750**

Total $= \frac{1000}{10}$

Final marks $= 100$

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1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n}(6r+1) = n(3n+4)$ for all $n \in \mathbb{Z}^+$.

For $n=1$, L.H.S. = 6+1=7 and

R.H.S. = 1(3+4) = 7

Hence, the result is true for $n=1$. (5)

For verifying the result for $n=1$

Let k be any positive integer and suppose that the result is true for $n=k$.

i.e $\sum_{r=1}^{k}(6r+1) = k(3k+4)$. (5)

For writing the statement for $n=k$

Now, $\sum_{r=1}^{k+1}(6r+1) = \sum_{r=1}^{k}(6r+1) + \{6(k+1)+1\}$

$= k(3k+4)+6k+7$ (5)

For substituting "$n=k$ result" in "$n=k+1$"

$= 3k^2+10k+7$

$= (k+1)(3k+7)$. (5)

$= (k+1)[3(k+1)+4]$.

$(k+1)(3k+7)$ or equivalent seen

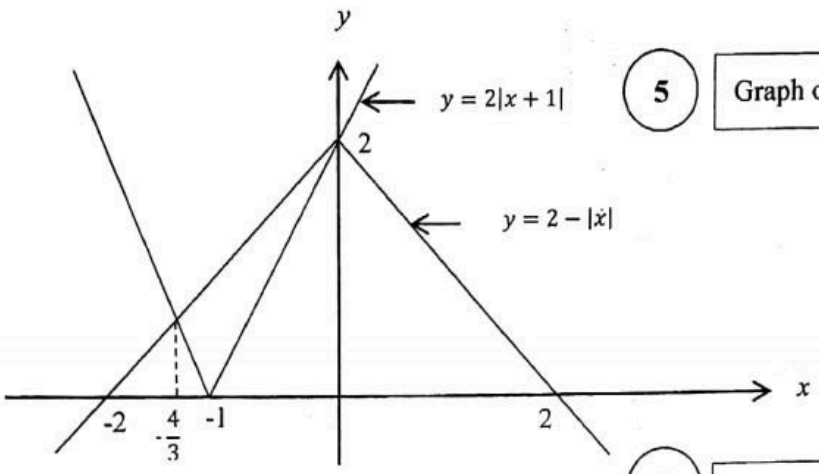
Hence, if the result is true for $n=k$, it is also true for $n=k+1$. We have already proved that the result is true for $n=1$.

Hence, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$ (5)

conclusion with the "Principle of Mathematical Induction". (Given only if all the other steps are correct.)

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2. Sketch the graphs of $y=2|x+1|$ and $y=2-|x|$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $2|x+2|+|x|\leq 4$.



5 Graph of $y=2|x+1|$

5 Graph of $y=2-|x|$

(To earn all 10 marks, the common intersection point on the y - axis must be seen; otherwise only 5)

The $x-$ coordinate of one point of intersection is $x=0$.

The $x-$ coordinate of the other point of intersection is given by $-2(x+1)=2+x$ for $x<-1$.

This gives $x=-\frac{4}{3}$. (5)

$x=0$ and $x=-\frac{4}{3}$ seen

Let $t=\frac{x}{2}$. (5)

$t=\frac{x}{2}$ substitution or equivalent

Then the given inequality becomes

$$2|2t+2|+|2t|\leq 4.$$

It is equivalent to

$$2|t+1|\leq 2-|t|.$$

From the graphs, we have

$$-\frac{4}{3}\leq t\leq 0.$$

$$\therefore\quad -\frac{8}{3}\leq x\leq 0.$$ (5)

correct solution seen

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Aliter 1:

For the graphs (5) + (5), as before.

Case (i) $x \leq -2$:

Then, $2|x+2|+|x| \leq 4$ is equivalent to $-2(x+2)-x \leq 4$.

$\therefore -\frac{8}{3} \leq x$.

Hence, in this case, solutions are the values of x satisfying $-\frac{8}{3} \leq x \leq -2$.

Case (ii) $-2 < x \leq 0$:

Then, $2|x+2|+|x| \leq 4$ is equivalent to $2(x+2)-x \leq 4$.

$\therefore x \leq 0$.

Hence, in this case, the solutions are the values of x satisfying $-2 < x \leq 0$.

Case (iii) $x > 0$:

Then, $2|x+2|+|x| \leq 4$ is equivalent to $2(x+2)+x \leq 4$

$\therefore x \leq 0$

Hence, in this case, there are no solutions.

All 3 cases with correct solutions	10
only 2 cases with correct solutions	5

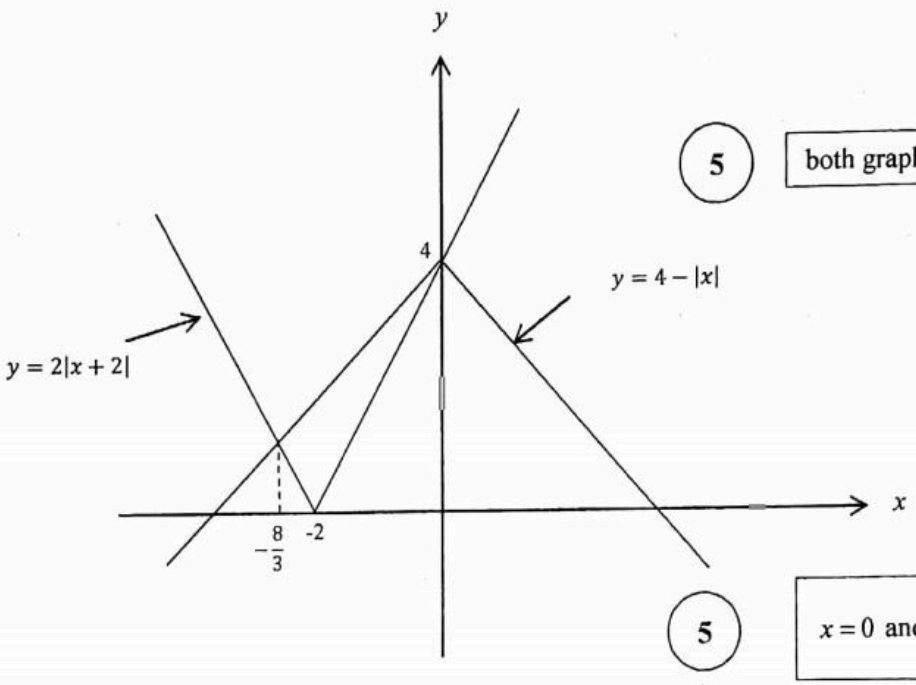
$\therefore$ The solutions of the given inequality are the values of x satisfying $-\frac{8}{3} \leq x \leq 0$. (5)

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Aliter 2:

For the graphs (5) + (5), as before.

$2|x+2|+|x| \le 4$ is equivalent to $2|x+2| \le 4-|x|$.



5 — both graphs

5 — $x=0$ and $x=-\frac{8}{3}$ seen

From the graphs,

$2|x+2| \le 4-|x|$

$\Leftrightarrow -\frac{8}{3} \le x \le 0$ (5)

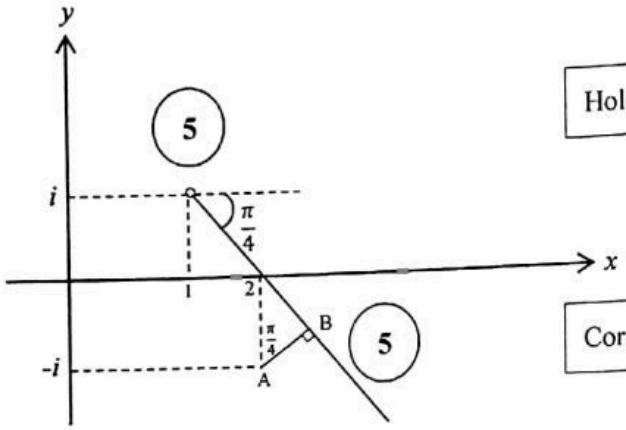
correct solution seen

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3. Sketch, in an Argand diagram, the locus of the points that represent complex numbers z satisfying $\text{Arg}(z-1-i)=-\frac{\pi}{4}$.

Hence or otherwise, show that the minimum value of $|z-2+i|$ satisfying $\text{Arg}(iz+1-i)=\frac{\pi}{4}$ is $\frac{1}{\sqrt{2}}$.

$$\text{Arg}\big(z-(1+i)\big)=-\frac{\pi}{4}$$



5

Hole at $1+i$

Correct half line

$$\text{Arg}\big(i(z-i-1)\big)=\frac{\pi}{4}$$

$$\therefore \text{Arg}\,i+\text{Arg}\big(z-(1+i)\big)=\frac{\pi}{4}$$

$$\therefore \text{Arg}\big(z-(1+i)\big)=-\frac{\pi}{4}$$ 5

Breaking the argument of the product to a sum, and using $\text{Arg}\,i=\frac{\pi}{2}$.

Now, min $|z-(2-i)|=AB$ 5

Recognising the minimum distance

$$=1\cdot\cos\frac{\pi}{4}$$

$$=\frac{1}{\sqrt{2}}$$ 5

Work leading to the answer

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Aliter:

For the diagram (5) + (5) as before.

Let $z = x + iy$

Then $\frac{\pi}{4} = Arg(iz + 1 - i) = Arg\left(1 - y + i(x - 1)\right)$

$\therefore \quad x - 1 = (1)(1 - y)$

$\Rightarrow \quad x + y = 2.$ (5)

Writing the given information as a relation of x and y.

Now $|z - 2 + i| = |x + iy - 2 + i|$

$= |(x - 2) + i(y + 1)|$

$= |y + i(y + 1)| \qquad (\because x - 2 = y)$

$= \sqrt{y^2 + (y + 1)^2}$

$= \sqrt{2\left(y + \frac{1}{2}\right)^2 + \frac{1}{2}}$ (5)

Writing the absolute value as a complete square of x or y.

$\geq \frac{1}{\sqrt{2}}$, since $2\left(y + \frac{1}{2}\right)^2 \geq 0$; $\quad (= 0 \text{ when } y = -\frac{1}{2})$.

$\therefore \min|z - 2 + i| = \frac{1}{\sqrt{2}}$ (5)

Work leading to the answer.

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4. Let $k > 0$. It is given that the coefficient of x^7 in the binomial expansion of $\left(x^2 + \frac{k}{x}\right)^{11}$ and the coefficient of x^{-7} in the binomial expansion of $\left(x - \frac{1}{x^2}\right)^{11}$ are equal. Show that $k = 1$.

$k > 0$. For $\left(x^2 + \frac{k}{x}\right)^{11}$;

$T_{r+1} = {}^{11}C_r (x^2)^{11-r} \left(\frac{k}{x}\right)^r = {}^{11}C_r x^{22-3r} k^r$

$22 - 3r = 7 \Rightarrow r = 5$ (5)

Correct value of r

$\therefore$ The coefficient of $x^7 = {}^{11}C_5 \; k^5$ (5)

Correct coefficient

For $\left(x - \frac{1}{x^2}\right)^{11}$; $T_{r+1} = {}^{11}C_r \; x^{11-r} \; (-1)^r \left(\frac{1}{x^2}\right)^r = (-1)^r \, {}^{11}C_r \; x^{11-3r}$

$11 - 3r = -7 \Rightarrow \; r = 6$ (5)

Correct value of r

$\therefore$ The coefficient of $x^{-7} = {}^{11}C_6$ (5)

Correct coefficient

Then, ${}^{11}C_6 = {}^{11}C_5 \; k^5$ gives $k = 1$, as ${}^{11}C_6 = {}^{11}C_5$. (5)

Work leading to the answer

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5. Show that $\lim_{x\to 0} \frac{\tan 2x - \sin 2x}{x^2\left(\sqrt{1+x}-\sqrt{1-x}\right)} = 4$.

$$\lim_{x\to 0} \frac{\tan 2x - \sin 2x}{x^2(\sqrt{1+x}-\sqrt{1-x})}$$

$$= \lim_{x\to 0} \frac{\frac{\sin 2x}{\cos 2x} - \sin 2x}{x^2(\sqrt{1+x}-\sqrt{1-x})} \times \frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)}{\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$ (5) Multiplying by the conjugate

$$= \lim_{x\to 0} \frac{\sin 2x}{2x} \times \frac{(1-\cos 2x)}{x^2\cos 2x} \times \left(\sqrt{1+x}+\sqrt{1-x}\right)$$

$$= \left(\lim_{2x\to 0} \frac{\sin 2x}{2x}\right) \times \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right)^2 \times \left(\lim_{x\to 0} \frac{1}{\cos 2x}\right) \times \lim_{x\to 0} \left(\sqrt{1+x}+\sqrt{1-x}\right)$$

$$= \underset{(5)}{1} \times \underset{(5)}{2} \times \underset{(5)}{1} \times \underset{(5)}{2}$$

Each limit (5)

$$= 4.$$

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Alter 1:

$$\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^2\left(\sqrt{1+x} - \sqrt{1-x}\right)}$$

$$= \lim_{x \to 0} \frac{\sin 2x(1-\cos 2x)}{x^2 \cos 2x} \cdot \frac{1}{\sqrt{1+x} - \sqrt{1-x}} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

(5) multiplying by the conjugate

$$= \lim_{x \to 0} \frac{\sin 2x(1-\cos^2 2x)}{x^2 \cos 2x(1+\cos 2x)} \cdot \frac{\sqrt{1+x} + \sqrt{1-x}}{2x}$$

$$= 4\left(\lim_{2x \to 0} \frac{\sin 2x}{2x}\right)^3 \left(\lim_{x \to 0} \frac{1}{\cos 2x}\right)\left(\lim_{x \to 0} \frac{1}{1+\cos 2x}\right)\left(\lim_{x \to 0} \sqrt{1+x} + \sqrt{1+x}\right)$$

Each limit (5)

$$= 4 \times 1 \times 1 \times \frac{1}{2} \times 2$$

(5) (5) (5) (5)

$$= 4.$$

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Aliter 2:

$$\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^2\left(\sqrt{1+x} - \sqrt{1-x}\right)}$$

5 multiplying by the conjugate

$$= \lim_{x \to 0} \frac{1}{x^2} \times \left(\frac{2\tan x}{1-\tan^2 x} - \frac{2\tan x}{1+\tan^2 x}\right) \frac{1}{\sqrt{1+x} - \sqrt{1-x}} \times \frac{\left(\sqrt{1+x} + \sqrt{1-x}\right)}{\left(\sqrt{1+x} + \sqrt{1-x}\right)}$$

$$= \lim_{x \to 0} \frac{2\tan x(2\tan^2 x)}{x^2(1-\tan^4 x)} \cdot \frac{\sqrt{1+x} + \sqrt{1-x}}{2x}$$

$$= 2\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^3 \left(\lim_{x \to 0} \frac{1}{\cos x}\right)^3 \left(\lim_{x \to 0} \frac{1}{1-\tan^4 x}\right)\left(\lim_{x \to 0} \sqrt{1+x} + \sqrt{1-x}\right)$$

$= 2 \times 1 \times 1 \times 1 \times 2$

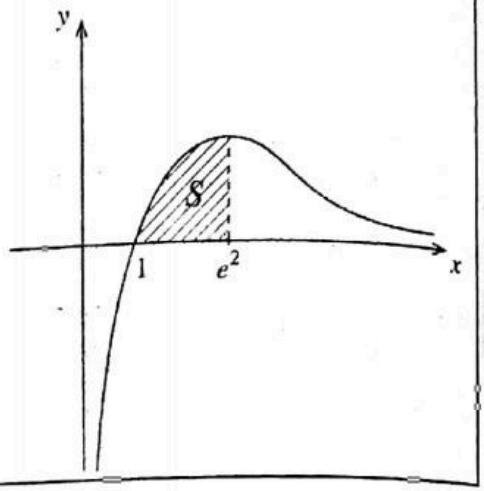
$= 4.$ (5) (5) (5) (5)

Each limit 5

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6. Let S be the region enclosed by the curves $y = \frac{\ln x}{\sqrt{x}}$, $y = 0$ and $x = e^2$. Show that the area of S is 4 square units.

The region S is rotated about the x-axis through 2π radians. Show that the volume of the solid thus generated is $\frac{8\pi}{3}$.



Area of $S = \int_1^{e^2} \frac{\ln x}{\sqrt{x}}\, dx$ (5)

Setting up the integral for S

$= (\ln x).2x^{\frac{1}{2}}\Big|_1^{e^2} - \int_1^{e^2} 2x^{\frac{1}{2}} \times \frac{1}{x} dx$ (5)

Integration by parts or equivalent

$= 4e - 2\int_1^{e^2} x^{-\frac{1}{2}} dx$

$= 4e - \left(2\sqrt{x}2\right)\Big|_1^{e^2}$

$= 4e - 4e + 4$

$= 4$ (5)

Work leading to the answer

The volume required $= \int_1^{e^2} \pi \left(\frac{\ln x}{\sqrt{x}}\right)^2 dx$ (5)

Setting up the integral for the volume

$= \pi \int_1^{e^2} \frac{(\ln x)^2}{x} dx$

$= \pi \frac{(\ln x)^3}{3}\Big|_1^{e^2}$

$= \frac{8\pi}{3}$. (5)

Work leading to the answer

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7. Show that the equation of the tangent line to the rectangular hyperbola parametrically given by $x = ct$ and $y = \frac{c}{t}$ for $t \neq 0$, at the point $P \equiv \left(cp, \frac{c}{p}\right)$ is given by $x + p^2y = 2cp$.

The normal line to this hyperbola at P meets the hyperbola again at another point $Q \equiv \left(cq, \frac{c}{q}\right)$.

Show that $p^3q = -1$.

$\frac{dx}{dt} = c$ and $\frac{dy}{dt} = -\frac{c}{t^2}$. $(t \neq 0.)$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{c}{t^2}}{c} = -\frac{1}{t^2}$$ (5)

$\frac{dy}{dx}$ in terms of t

$\therefore$ The gradient of the tangent at $P = -\frac{1}{t^2}\Big|_{t=p} = -\frac{1}{p^2}$

$\therefore$ The equation of the tangent at P:

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$

$\therefore x + p^2y = 2cp.$ (5)

Work leading to the answer

The gradient of the normal at $P = p^2$.

$\therefore$ Equation of the normal at P; $y - \frac{c}{p} = p^2(x - cp)$ (5)

Equation of the normal

$Q \equiv (cq, \frac{c}{q})$ is on this line;

$$\therefore \frac{c}{q} - \frac{c}{p} = p^2(cq - cp) \Rightarrow c(p - q) = -p^3qc(p - q)$$ (5)

For the substitution

Since P and Q are distinct points, we have $p \neq q$.

$p^3q = -1.$ (5)

Work leading to the answer

Aliter: (For the last part)

Gradient of PQ = Gradient of the normal line at P (5)

For the condition

$$\therefore \frac{\frac{c}{q} - \frac{c}{p}}{cq - cp} = p^2$$ (5) $(\because p \neq q, c \neq o)$

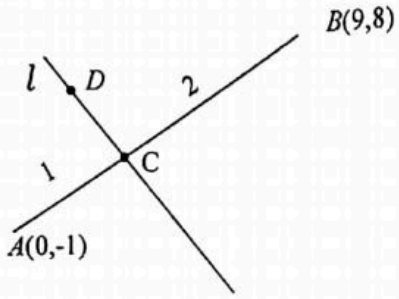
For the substitutions

$\therefore p^3q = -1$ (5)

Work leading to the answer

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8. Let $A \equiv (0,-1)$ and $B \equiv (9,8)$. The point C lies on AB such that $AC:CB = 1:2$. Show that the equation of the straight line l through C perpendicular to AB is $x+y-5=0$.
Let D be the point on l such that AD is parallel to the straight line $y = 5x+1$. Find the coordinates of D.



$$C \equiv \left(\frac{2(0)+1(9)}{2+1}, \frac{2(-1)+1(8)}{2+1}\right)$$

$\equiv (3,2)$ **5**

Coordinates of C

The gradient of $AB = 1$.

The gradient of $l = -1$ **5**

For the gradient of l

$\therefore$ Equation of l:

$$y-2 = -1(x-3)$$

i.e. $x+y-5=0$. ---------- (1) **5**

Work leading to the equation of l

Equation of AD; $y-(-1) = 5(x-0)$ **5**

Work leading to the coordinates of D

i.e. $y+1 = 5x$ ---------- (2)

Solving (1) and (2)

$\therefore$ $D \equiv (1,4)$. **5**

$D \equiv (1,4)$ seen

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9. Show that the straight line $x + 2y = 3$ intersects the circle $S \equiv x^2 + y^2 - 4x + 1 = 0$ at two distinct points.

Find the equation of the circle passing through these two points and the centre of the circle $S = 0$.

$S \equiv x^2 + y^2 - 4x + 1 = 0$

Let ℓ: $x + 2y - 3 = 0$.

On ℓ, $x = 3 - 2y$;

$$(3-2y)^2 + y^2 - 4(3-2y) + 1 = 0$$

$$\therefore \quad 5y^2 - 4y - 2 = 0 \quad (5)$$

Forming a quadratic

Discriminant of this quadratic $\Delta = 16 + 4(5)(2)$ (5)

Writing the discriminant

$\therefore$ Since $\Delta > 0$, the line $x + 2y = 3$ intersects S at two distinct points. (5)

For $\Delta > 0$

The equation of the required circle can be written as

$$x^2 + y^2 - 4x + 1 + \lambda(x + 2y - 3) = 0,$$ (5)

For the λ form

where $\lambda \in \mathbb{R}$.

This circles passes through (2,0), we have

$$4 - 8 + 1 + \lambda(2 - 3) = 0$$

$$\therefore \lambda = -3 \quad (5)$$

$\lambda = -3$ seen

$\therefore$ Equation of the required circle is

$$x^2 + y^2 - 4x + 1 + (-3)(x + 2y - 3) = 0$$

i.e. $x^2 + y^2 - 7x - 6y + 10 = 0$.

Aliter:

5 5

Perpendicular distance from the centre < radius

Comparison (5)

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10. Express $2\cos^2 x + 2\sqrt{3}\sin x\cos x - 1$ in the form $R\cos(2x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Hence, solve the equation $\cos^2 x + \sqrt{3}\sin x\cos x = 1$.

$2\cos^2 x + 2\sqrt{3}\sin x\cos x - 1$

$= 2\cos^2 x - 1 + \sqrt{3}(2\sin x\cos x)$

$= \cos 2x + \sqrt{3}\sin 2x$ (5)

Writing the expression using $\cos 2x$ and $\sin 2x$

$= 2\left[\frac{1}{2}\cos 2x + \frac{\sqrt{3}}{2}\sin 2x\right]$

$= 2\cos\left(2x - \frac{\pi}{3}\right)$

$R = 2$ seen

Here $R = 2$ (5) and $\alpha = \frac{\pi}{3}$ (5)

$\alpha = \frac{\pi}{3}$ seen

The equation $\cos^2 x + \sqrt{3}\sin x = 1$ is equivalent to

$2\cos^2 x + 2\sqrt{3}\sin x\cos x - 1 = 1$.

$\therefore \quad 2\cos\left(2x - \frac{\pi}{3}\right) = 1$

Hence $\cos\left(2x - \frac{\pi}{3}\right) = \frac{1}{2}$ (5)

$\cos\left(2x - \frac{\pi}{3}\right) = \frac{1}{2}$ seen

$\therefore \quad 2x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3}; \ n \in \mathbb{Z}$

$\therefore \quad x = n\pi + \frac{\pi}{6} \pm \frac{\pi}{6}; \ n \in \mathbb{Z}$ (5)

Correct solution seen

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11. (*a*) Let $k > 1$. Show that the equation $x^2 - 2(k+1)x + (k-3)^2 = 0$ has real distinct roots.
Let α and β be these roots. Write down $\alpha+\beta$ and $\alpha\beta$ in terms of k, and find the values of k such that both α and β are positive.
Now, let $1 < k < 3$. Find the quadratic equation whose roots are $\frac{1}{\sqrt{\alpha}}$ and $\frac{1}{\sqrt{\beta}}$, in terms of k.

(*b*) Let $f(x) = 2x^3 + ax^2 + bx + 1$ and $g(x) = x^3 + cx^2 + ax + 1$, where $a, b, c \in \mathbb{R}$. It is given that the remainder when $f(x)$ is divided by $(x-1)$ is 5, and that the remainder when $g(x)$ is divided by $x^2 + x - 2$ is $x + 1$. Find the values of a, b and c.
Also, with these values for a, b and c, show that $f(x) - 2g(x) \leq \frac{13}{12}$ for all $x \in \mathbb{R}$.

(a)

Let Δ be the discriminant of $x^2 - 2(k+1)x + (k-3)^2 = 0$.

Then $\Delta = 4(k+1)^2 - 4(k-3)^2$ (5)

$= 4(k+1+k-3)(k+1-k+3)$

$= 32(k-1)$. (5)

Since $k > 1$, we have $\Delta > 0$. (5)

$\therefore$ The given equation has real distinct roots. (5)

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$\alpha + \beta = 2(k+1)$ and $\alpha\beta = (k-3)^2$ (5) + (5)

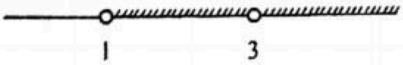
For α and β both to be positive,

we must have $\alpha + \beta > 0$ and $\alpha\beta > 0$. (10)

Since $k > 1$, we have $\alpha + \beta = 2(k+1) > 0$ (5)

and $\alpha\beta = (k-3)^2 > 0$ if and only if $k \neq 3$. (10)

$\therefore$ The required values of k are $1 < k < 3$ or $k > 3$.



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Now let $1<k<3$. Note that $\alpha>0$ and $\beta>0$.

The equation whose roots are $\frac{1}{\sqrt{\alpha}}$ and $\frac{1}{\sqrt{\beta}}$ is $\left(x-\frac{1}{\sqrt{\alpha}}\right)\left(x-\frac{1}{\sqrt{\beta}}\right)=0$. (5)

i.e. $x^2-\left(\frac{1}{\sqrt{\alpha}}+\frac{1}{\sqrt{\beta}}\right)x+\frac{1}{\sqrt{\alpha\beta}}=0$. (5)

i.e. $\sqrt{\alpha\beta}x^2-\left(\sqrt{\alpha}+\sqrt{\beta}\right)x+1=0$. (5)

Note that $\sqrt{\alpha\beta}=\sqrt{(k-3)^2}=|k-3|=3-k$ $(\because 1<k<3)$

Also, $\left(\sqrt{\alpha}+\sqrt{\beta}\right)^2=\alpha+\beta+2\sqrt{\alpha\beta}$ (5)

$=2(k+1)+2(3-k)$ (5)

$=8.$ (5)

$\therefore \sqrt{\alpha}+\sqrt{\beta}=2\sqrt{2}$ (5) $\left(\because \sqrt{\alpha}+\sqrt{\beta}>0.\right)$

$\therefore$ The required equation is $(3-k)x^2-2\sqrt{2}x+1=0$ (5)

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(b)

$f(x)=2x^3+ax^2+bx+1$ and

$g(x)=x^3+cx^2+ax+1$

Since the remainder when $f(x)$ is divided by $(x-1)$ in 5, by the Remainder Theorem, $f(1)=5$. (5)

$\therefore\ a+b+3=5$

$a+b=2.$ (5) ---------- (1)

Since, the remainder when $g(x)$ in divided by x^2+x-2 is $x+1$, we have

$g(x)=x^3+cx^2+ax+1=(x^2+x-2)(x+\lambda)+x+1$ for $\lambda\in\mathbb{R}$. (5)

$\left((x^0)\right)$; $\quad 1=-2\lambda+1$ gives $\lambda=0$.

$\therefore g(x)=x(x^2+x-2)+x+1$ (5) (5)

$=x^3+x^2-x+1.$ Hence $c=1$ and $a=-1$.

Now by (1); $b=3$. (5)

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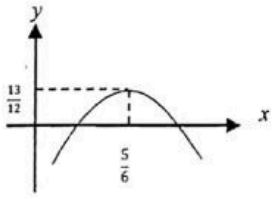
$$f(x)-2g(x)=2x^3-x^2+3x+1-2(x^3+x^2-x+1)$$

$$=-3x^2+5x-1 \quad (5)$$

$$=-3\left[\left(x-\frac{5}{6}\right)^2-\frac{25}{36}+\frac{1}{3}\right]$$

$$=-3\left[\left(x-\frac{5}{6}\right)^2-\frac{13}{36}\right] \quad (5)$$

$$\leq -3 \times \left(\frac{-13}{36}\right), \text{ since } \left(x-\frac{5}{6}\right)^2 \geq 0. \quad (5)$$



$$\therefore\ f(x)-2g(x)\leq\frac{13}{12}. \quad (5)$$

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12. (*a*) It is required to form a 4-digit number consisting of 4 digits taken from the 10 digits given below:

1, 1, 1, 2, 2, 3, 3, 4, 5, 5

Find the number of different such 4-digit numbers that can be formed

(i) if all 4 digits chosen are different,

(ii) if any 4 digits can be chosen.

(*b*) Let $U_r = \dfrac{-16r^3+12r^2+40r+9}{5(2r+1)^2(2r-1)^2}$ for $r \in \mathbb{Z}^+$.

Determine the values of the real constants A and B such that $U_r = \dfrac{A(r-1)}{(2r+1)^2} - \dfrac{(r-B)}{(2r-1)^2}$ for $r \in \mathbb{Z}^+$.

Hence, find $f(r)$ such that $\dfrac{1}{5^{r-1}}U_r = f(r) - f(r-1)$ for $r \in \mathbb{Z}^+$, and

show that $\displaystyle\sum_{r=1}^{n} \frac{1}{5^{r-1}}U_r = 1 + \frac{n-1}{5^n(2n+1)^2}$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\displaystyle\sum_{r=1}^{\infty} \frac{1}{5^{r-1}}U_r$ is convergent and find its sum.

(a)

1, 1, 1, 2, 2, 3, 3, 4, 5, 5

(i) Four different digits out of 1,2,3,4 and 5

$= {}^5P_4$ (5)

$= 5!$ (5)

$= 120$ (5)

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(ii) four digit numbers can be formed by

	number of different such 4 digit numbers
four different digits	${}^5P_4 = 120$
only one digit is repeated twice and the other two are different	**10** **5** ${}^4C_1 \times {}^4C_2 \times \frac{4!}{2!} = 288$
Two digits repeated twice	**10** ${}^4C_2 \times \frac{4!}{2!2!} = 36$ **5**
one digits repeated thrice	**10** ${}^1C_1 \times {}^4C_1 \times \frac{4!}{3!} = 16$ **5**

The required number of ways $= 120 + 288 + 36 + 16$ **5**

$= 460$ **5**

55

(b)

For $r \in \mathbb{Z}^+$

$$U_r = \frac{-16r^3 + 12r^2 + 40r + 9}{5(2r+1)^2(2r-1)^2}$$

$$U_r = \frac{A(r-1)}{(2r+1)^2} - \frac{(r-B)}{(2r-1)^2} = \frac{A(r-1)(2r-1)^2 - (r-B)(2r+1)^2}{(2r+1)^2(2r-1)^2}$$

$$\therefore -16r^3 + 12r^2 + 40r + 9 \equiv 5A(r-1)(4r^2 - 4r + 1) - 5(r-B)(4r^2 + 4r + 1)$$

Comparing coefficients of powers of r:

$r^3 : -16 = 5A(4) - 20$

$r^2 : 12 = 5A(-8) - 5(-4B + 4)$

$r^1 : 40 = 25A - 5(1 - 4B)$

$r^0 : 9 = -5A + 5B$ **10**

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These give us $A=\frac{1}{5}$ and $B=2$. (5) (5)

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$\therefore U_r=\frac{r-1}{5(2r+1)^2}-\frac{r-2}{(2r-1)^2}$ (5)

$\therefore \frac{1}{5^{r-1}}U_r=\frac{r-1}{5^r(2r+1)^2}-\frac{r-2}{5^{r-1}(2r-1)^2}$ (5)

and hence,

$\frac{1}{5^{r-1}}U_r=f(r)-f(r-1)$, where $f(r)=\frac{r-1}{5^r(2r+1)^2}$. (10)

$r=1;\quad \frac{1}{5^0}U_1=f(1)-f(0)$

$r=2;\quad \frac{1}{5}U_2=f(2)-f(1)$ } (5)

$\vdots \qquad \vdots$

$r=n-1;\quad \frac{1}{5^{n-2}}U_{n-1}=f(n-1)-f(n-2)$

$r=n;\quad \frac{1}{5^{n-1}}U_n=f(n)-f(n-1)$ } (5)

$\sum_{r=1}^{n}\frac{1}{5^{r-1}}U_r=f(n)-f(0)$ (5)

$=\frac{n-1}{5^n(2n+1)^2}-(-1)$ (5)

$=1+\frac{n-1}{5^n(2n+1)^2}$ for $n\in\mathbb{Z}^+$ (5)

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$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{5^{r-1}}U_r = \lim_{n\to\infty}\left(1+\frac{n-1}{5^n(2n+1)^2}\right)$$

(5) $= 1$. (5)

$\therefore$ the infinite series $\sum_{r=1}^{\infty}\frac{1}{5^{r-1}}U_r$ is convergent and the sum is 1. (5)

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13.(*a*) Let $\mathbf{A} = \begin{pmatrix} a & 0 & 3 \\ 0 & a & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, where $a \in \mathbb{R}$.

Also, let $\mathbf{C} = \mathbf{AB}^T$. Find $\mathbf{C}$ in terms of a, and show that $\mathbf{C}^{-1}$ exists for all $a \neq 0$.

Write down $\mathbf{C}^{-1}$ in terms of a, when it exists.

Show that if $\mathbf{C}^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 9 \\ -11 \end{pmatrix}$, then $a = 2$.

With this value for a, find the matrix $\mathbf{D}$ such that $\mathbf{DC} - \mathbf{C}^T\mathbf{C} = 8\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Let $z_1 = 1+\sqrt{3}i$ and $z_2 = 1+i$. Express $\frac{z_1}{z_2}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

Also, express each of the complex numbers z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$, and **hence**, show that $\frac{z_1}{z_2} = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$.

Deduce that $\cos\left(\frac{\pi}{12}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

(*c*) Let $n \in \mathbb{Z}^+$ and $\theta \neq 2k\pi \pm \frac{\pi}{2}$ for $k \in \mathbb{Z}$.

Using De Moivre's theorem, show that $(1+i\tan\theta)^n = \sec^n\theta(\cos n\theta + i\sin n\theta)$.

Hence, obtain a similar expression for $(1-i\tan\theta)^n$, and

show that $(1+i\tan\theta)^n + (1-i\tan\theta)^n = 2\sec^n\theta\cos n\theta$.

Deduce that $z = i\tan\left(\frac{\pi}{10}\right)$ is a solution of $(1+z)^{25} + (1-z)^{25} = 0$.

(*a*) (5)

$$C = AB^T = \begin{pmatrix} a & 0 & 3 \\ 0 & a & 1 \end{pmatrix}\begin{pmatrix} a & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a^2+3 & a+3 \\ a+1 & 1 \end{pmatrix}$$ (10)

All 4 entries correct (10)

only 3 correct (5)

$$|C| = (a^2+3) - (a+1)(a+3) = -4a$$

$\neq 0$ ($\because a \neq 0$) (5)

$\therefore$ C^{-1} exists for all $a \neq 0$. (5)

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For $a \neq 0, C^{-1} = -\frac{1}{4a}\begin{pmatrix} 1 & -(a+3) \\ -(a+1) & a^2+3 \end{pmatrix}$ (10)

All 4 entries correct (10)

only 3 correct (5)

10

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$C^{-1}\begin{pmatrix}1\\2\end{pmatrix} = \frac{1}{4a}\begin{pmatrix}-1 & a+3\\ a+1 & -a^2-3\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \frac{1}{4a}\begin{pmatrix}2a+5\\ -2a^2+a-5\end{pmatrix}$ (10)

All 2 entries correct (10)
only 1 correct (5)

$\therefore \frac{1}{4a}\begin{pmatrix}2a+5\\ -2a^2+a-5\end{pmatrix} = \frac{1}{8}\begin{pmatrix}9\\ -11\end{pmatrix}$

$\frac{2a+5}{4a} = \frac{9}{8}$ and $\frac{-2a^2+a-5}{4a} = -\frac{11}{8}$ (5)

These two equations give us $a = 2$. (5)

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When $a = 2$, $C = \begin{pmatrix}7 & 5\\ 3 & 1\end{pmatrix}$ and $C^{-1} = -\frac{1}{8}\begin{pmatrix}1 & -5\\ -3 & 7\end{pmatrix}$. (5)

$DC - C^TC = 8I$ is equivalent to $D - C^T = 8IC^{-1}$. (5)

(5)

$\therefore D = C^T + 8C^{-1} = \begin{pmatrix}7 & 3\\ 5 & 1\end{pmatrix} + 8\left(-\frac{1}{8}\right)\begin{pmatrix}1 & -5\\ -3 & 7\end{pmatrix} = \begin{pmatrix}6 & 8\\ 8 & -6\end{pmatrix}$. (5)

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(b)

$\frac{z_1}{z_2} = \frac{1+\sqrt{3}i}{1+i} \times \frac{1-i}{1-i} = \frac{1}{2}(1+\sqrt{3}i)(1-i) = \underbrace{\frac{1+\sqrt{3}}{2}}_{x} + i\underbrace{\left(\frac{\sqrt{3}-1}{2}\right)}_{y}$ (5)

(5)

10

$z_1 = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (5)

(5)

$z_2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (5)

(5)

$\therefore \frac{z_1}{z_2} = \frac{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} = \sqrt{2}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ (10)

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Equating real parts,

$$\sqrt{2}\cos\frac{\pi}{12}=\frac{1+\sqrt{3}}{2}$$

$$\therefore \cos\frac{\pi}{12}=\frac{1+\sqrt{3}}{2\sqrt{2}}$$ (5)

05

(c)

For $n\in\mathbb{Z}^+$ and $\theta\neq 2k\pi\pm\frac{\pi}{2}$ for $k\in\mathbb{Z}$,

$$(1+i\tan\theta)^n=\frac{1}{\cos^n\theta}(\cos\theta+i\sin\theta)^n$$ (5)

$$=\sec^n\theta(\cos n\theta+i\sin n\theta)$$ ---------- (1) (5)

10

$$(1-i\tan\theta)^n=\left(1+i\tan(-\theta)\right)^n$$

$$=\sec^n(-\theta)\left[\cos n(-\theta)+i\sin n(-\theta)\right]$$

$$=\sec^n\theta(\cos n\theta-i\sin n\theta)$$ ---------- (2) (5)

(1) and (2) give us $(1+i\tan\theta)^n+(1-i\tan\theta)^n=2\sec^n\theta\cos n\theta$. (5)

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$z=i\tan\left(\frac{\pi}{10}\right)$ gives

$$(1+z)^{25}+(1-z)^{25}=\left(1+i\tan\left(\frac{\pi}{10}\right)\right)^{25}+\left(1-i\tan\left(\frac{\pi}{10}\right)\right)^{25}$$

$$=2\sec^{25}\left(\frac{\pi}{10}\right)\cos 25\left(\frac{\pi}{10}\right)$$ (5)

$$=0,\ \text{as}\ \cos 25\left(\frac{\pi}{10}\right)=\cos\frac{\pi}{2}=0.$$ (5)

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14. (a) Let $f(x)=\dfrac{4x+1}{x(x-2)}$ for $x \neq 0, 2$.

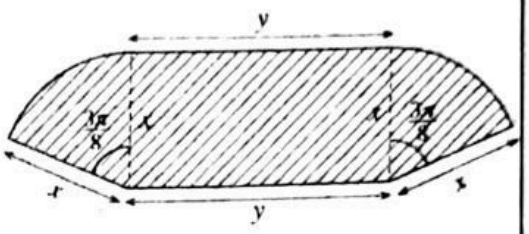
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=-\dfrac{2(2x-1)(x+1)}{x^2(x-2)^2}$ for $x \neq 0, 2$.

Hence, find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Sketch the graph of $y=f(x)$ indicating the asymptotes, x-intercept and the turning points.

Using this graph, find all real values of x satisfying the inequality $f(x)+|f(x)|>0$.

(b) The shaded region S of the adjoining figure shows a garden consisting of a rectangle and two sectors of a circle each subtending an angle $\frac{3\pi}{8}$ at the centre. Its dimensions, in metres, are shown in the figure. The area of S is given to be 36 m^2. Show that the perimeter p m of S is given by $p=2x+\dfrac{72}{x}$ for $x>0$ and that p is minimum when $x=6$.



(a)

For $x \neq 0,2$, $f(x)=\dfrac{4x+1}{x(x-2)}$

Then, $f'(x)=\dfrac{4x(x-2)-(4x+1)(x-2+x)}{x^2(x-2)^2}$ (20)

$=-\dfrac{2(2x^2+x-1)}{x^2(x-2)^2}$

$=-\dfrac{2(2x-1)(x+1)}{x^2(x-2)^2}$ for $x \neq 0,2$.

(5)

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Turning points:

$f'(x)=0 \quad \Leftrightarrow \quad x=-1 \quad \text{or} \quad x=\dfrac{1}{2}$ (5)

	$-\infty<x<-1$	$-1<x<0$	$0<x<\frac{1}{2}$	$\frac{1}{2}<x<2$	$2<x<\infty$
sign of $f'(x)$	(−)	(+)	(+)	(−)	(−)
$f(x)$ is	↘ decreasing	↗ increasing	↗ increasing	↘ decreasing	↘ decreasing
	(5)	(5)	(5)	(5)	(5)

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$\therefore f(x)$ is increasing on $[-1,0)$ and $\left(0,\frac{1}{2}\right]$

and decreasing on $(-\infty,-1]$, $\left[\frac{1}{2},2\right)$ and $(2,\infty)$.

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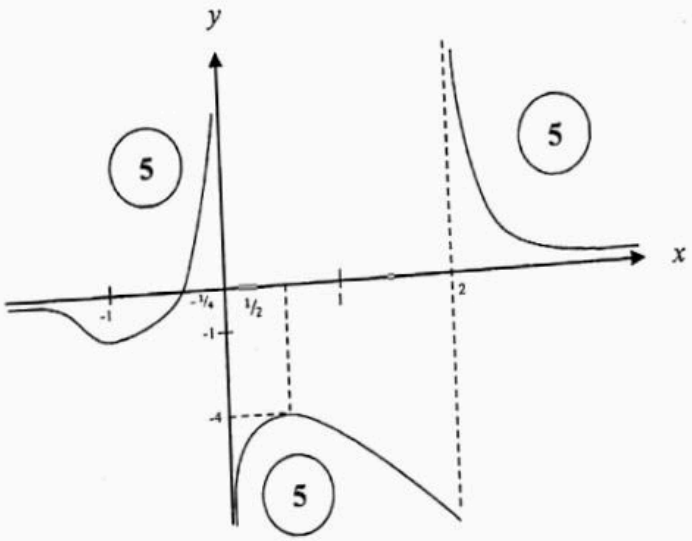
Turning points: $\left(\frac{1}{2},-4\right)$ is a local maximum. 5

$(-1,-1)$ is a local minimum. 5

x - intercept : $\left(-\frac{1}{4},0\right)$ 5

Horizontal asymptote: $\lim_{x\to\pm\infty} f(x)=0 \therefore y=0$ 5

Vertical asymptotes: $x=0$ and $x=2$. 5



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Note that $f(x)+|f(x)| = \begin{cases} 2f(x) & \text{if } f(x)\geq 0. \\ 0 & \text{if } f(x)<0. \end{cases}$

$\therefore f(x)+|f(x)|>0$ if and only if $f(x)>0$.

$\therefore$ The real values of satisfying $f(x)+|f(x)|>0$ is given by

$\Leftrightarrow -\frac{1}{4}<x<0$ or $x>2$. 5

10

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(b)

For $x > 0$;

Given: $36 = xy + \frac{3}{8}\pi x^2$ **10**

$\therefore \quad y = \frac{36}{x} - \frac{3}{8}\pi x \quad$ for $x > 0$

$p = 2x + 2y + 2\left(\frac{3}{8}\pi x\right)$ **10**

$= 2x + 2\left(\frac{36}{x} - \frac{3}{8}\pi x\right) + \frac{3}{4}\pi x$

$\therefore p = 2x + \frac{72}{x}$ **5**

$\frac{dp}{dx} = 2 - \frac{72}{x^2}; \; x > 0.$

5

$\frac{dp}{dx} = 0 \quad \Leftrightarrow \quad x = 6.$ **5**

For $0 < x < 6, \quad \frac{dp}{dx} < 0$ and

for $x > 6, \quad \frac{dp}{dx} > 0.$

$\therefore$ p is minimum when $x = 6.$ **5**

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15. (*a*) Find the values of the constants A, B and C such that

$x^4+3x^3+4x^2+3x+1=A(x^2+1)^2+Bx(x^2+1)+Cx^2$ for all $x\in\mathbb{R}$.

Hence, write down $\dfrac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}$ in partial fractions and

find $\displaystyle\int\frac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}\,dx$.

(*b*) Let $I=\displaystyle\int_0^{\frac{1}{2}}\sin^{-1}\left(\sqrt{x}\right)dx$. Show that $I=\frac{\pi}{24}-\frac{1}{2}\int_0^{\frac{1}{4}}\sqrt{\frac{x}{1-x}}\,dx$ and **hence**, evaluate I.

(*c*) Show that $\dfrac{d}{dx}\left(x\ln(x^2+1)+2\tan^{-1}x-2x\right)=\ln(x^2+1)$.

Hence, find $\displaystyle\int\ln(x^2+1)\,dx$ and show that $\displaystyle\int_0^1\ln(x^2+1)dx=\frac{1}{2}(\ln 4+\pi-4)$.

Using the result $\displaystyle\int_0^a f(x)dx=\int_0^a f(a-x)dx$, where a is a constant,

find the value of $\displaystyle\int_0^1\ln\left[(x^2+1)(x^2-2x+2)\right]dx$.

(*a*)

$x^4+3x^3+4x^2+3x+1=A(x^2+1)^2+Bx(x^2+1)+Cx^2$

$=A(x^4+2x^2+1)+B(x^3+x)+Cx^2$

Comparing coefficients of powers of x;

x^0 : $1=A$

x : $3=B$

x^2 : $4=2A+C$ (5) + (5)

x^3 : $3=B$

x^4 : $1=A$

$\therefore A=1$, $B=3$ and $C=2$. (5)

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$$\frac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}=\frac{1}{x}+\frac{3}{x^2+1}+\frac{2x}{(x^2+1)^2}$$ (10)

$$\int\frac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}=\int\frac{1}{x}\,dx+3\int\frac{1}{x^2+1}\,dx+2\int\frac{x}{(x^2+1)^2}\,dx.$$ (5)

$$=\ln|x|+3\tan^{-1}x-\frac{1}{x^2+1}+E,$$ where E is an arbitrary constant.

(5) (5) (5) (5)

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(*b*)

$$I=\int_0^{\frac{1}{4}}\sin^{-1}(\sqrt{x})dx$$

$$=x\sin^{-1}\left(\sqrt{x}\right)\Big|_0^{\frac{1}{4}}-\int_0^{\frac{1}{4}}x\frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}dx.$$ (10)

$$=\frac{1}{4}.\frac{\pi}{6}-\frac{1}{2}\int_0^{\frac{1}{4}}\sqrt{\frac{x}{1-x}}dx$$ (5)

$$=\frac{\pi}{24}-\frac{1}{2}\int_0^{\frac{1}{4}}\sqrt{\frac{x}{1-x}}dx$$ (5)

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Let $\sqrt{x}=\sin\theta \quad \Rightarrow dx=2\sin\theta\cos\theta d\theta$ (5)

$\theta=0$ when $x=0$

$\theta=\frac{\pi}{6}$ when $x=\frac{1}{4}$ (5)

$$\int_0^{\frac{1}{4}}\sqrt{\frac{x}{1-x}}dx=\int_0^{\frac{\pi}{6}}\frac{\sin\theta}{\cos\theta}\,2\sin\theta\cos\theta d\theta$$ (5)

$$=\int_0^{\frac{\pi}{6}}(1-\cos 2\theta)d\theta$$ (5)

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$= \left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{6}}$ (5)

$= \frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (5)

$\therefore I = \frac{\pi}{24} - \frac{1}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = -\frac{\pi}{24} + \frac{\sqrt{3}}{8} = \frac{3\sqrt{3} - \pi}{24}$.

(5)

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(c)

$\frac{d}{dx}(x\ln(x^2+1) + 2\tan^{-1}x - 2x)$

$= x\left(\frac{1}{x^2+1}\right)(2x) + \ln(x^2+1) + \frac{2}{1+x^2} - 2$ (10)

$= \ln(x^2+1) + \underbrace{\frac{2x^2 + 2 - 2(1+x^2)}{1+x^2}}_{=0}$

$= \ln(x^2+1)$. (5)

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$\int \ln(x^2+1)dx = x\ln(x^2+1) + 2\tan^{-1}x - 2x + C$, where C is an arbitrary constant. (5)

$\therefore \int_0^1 \ln(x^2+1)dx = \ln 2 + 2\left(\frac{\pi}{4}\right) - 2$ (5)

$= \ln 2 + \frac{\pi}{2} - 2$

$= \frac{1}{2}(2\ln 2 + \pi - 4)$

$= \frac{1}{2}(\ln 4 + \pi - 4)$ (5)

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$$\int_0^1 \ln\left[(x^2+1)(x^2-2x+2)\right]dx$$

$$= \int_0^1 \ln(x^2+1) + \int_0^1 \ln(x^2-2x+2)dx \quad (5)$$

Now $\int_0^1 \ln(x^2-2x+2)dx$

$$= \int_0^1 \ln\left((1-x)^2-2(1-x)+2\right)dx$$

$$= \int_0^1 \ln(x^2+1)dx \quad (5)$$

$$\therefore \int_0^1 \ln\left[(x^2+1)(x^2-2x+2)\right]dx = 2\int_0^1 \ln(x^2+1)dx$$

$$= \ln 4 + \pi - 4 \quad (5)$$

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16. Let $P \equiv (x_1, y_1)$ and l be the straight line given by $ax+by+c=0$. Show that the coordinates of any point on the line through the point P and perpendicular to l are given by (x_1+at, y_1+bt), where $t \in \mathbb{R}$.

Deduce that the perpendicular distance from P to l is $\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$.

Let l be the straight line $x+y-2=0$. Show that the points $A \equiv (0,6)$ and $B \equiv (3,-3)$ lie on opposite sides of l.

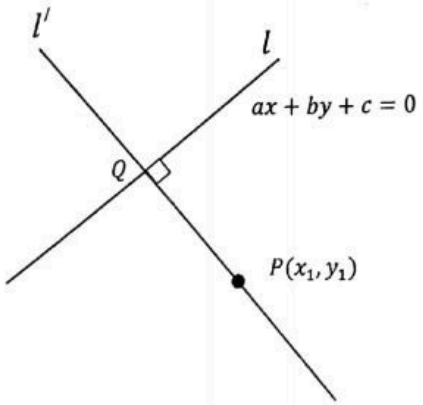
Find the acute angle between l and the line AB.

Find the equations of the circles S_1 and S_2 with centres at A and B, respectively, and touching l.

Let C be the point of intersection of l and the line AB. Find the coordinates of the point C.

Find also the equation of the other common tangent through C to S_1 and S_2.

Show that the equation of the circle that passes through the origin, bisects the circumference of S_1 and orthogonal to S_2 is $3x^2+3y^2-38x-22y=0$.



(Note that $a^2+b^2 \neq 0$)

The equation of l^1 : $y-y_1=\dfrac{b}{a}(x-x_1)$. (5)

$\therefore \dfrac{y-y_1}{b}=\dfrac{x-x_1}{a}=t$ (say) (5)

Then $x=x_1+at$, $y=y_1+bt$ (5)

(This is valid even when $a=0$ and $b\neq 0$ or when $a\neq 0$ and $b=0$.)

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Let $Q \equiv (x_2, y_2) \equiv (x_1 + at_1, y_1 + bt_1)$ be the point of intersection of l and l^1.

Since Q is on l, we have $a(x_1 + at_1) + b(y_1 + bt_1) + c = 0$.

$$\therefore t_1 = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}. \quad (5)$$

The perpendicular distance from P to $l = PQ$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (5)$$

$$= \sqrt{a^2 t_1^2 + b^2 t_1^2}$$

$$= \sqrt{a^2 + b^2}\,|t_1|. \quad (5)$$

$$= \sqrt{a^2 + b^2}.\ \frac{|ax_1 + by_1 + c|}{(a^2 + b^2)}$$

$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}. \quad (5)$$

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$\ell : x + y - 2 = 0$

$(0 + 6 - 2)(3 - 3 - 2) = -8 < 0$ (5)

(5)

$\therefore$ A and B lie on opposite sides of ℓ

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The gradient of $AB = -3$ (5)

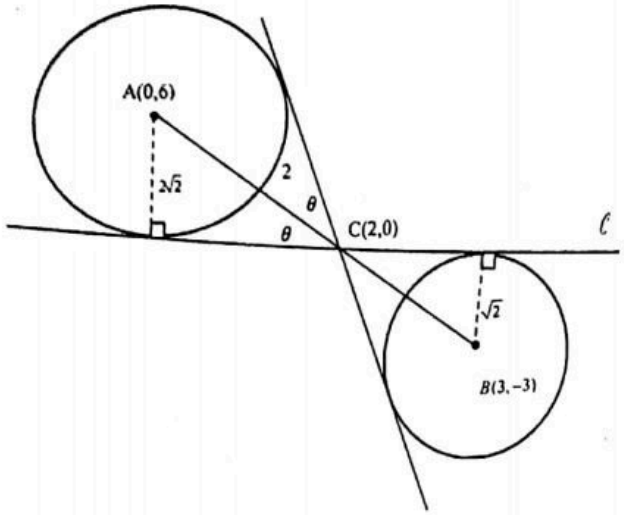
The acute angle between ℓ and AB

$$\tan\theta = \left|\frac{-1 - (-3)}{1 + (-1)(-3)}\right| \quad (5)$$

$$\therefore \theta = \tan^{-1}\left(\frac{1}{2}\right) \quad (5)$$

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(5) (5)

The radius of $S_1 = \frac{|0+6-2|}{\sqrt{2}} = 2\sqrt{2}$ and the radius of $S_2 = \frac{|3-3-2|}{\sqrt{2}} = \sqrt{2}$. (5) (5)

$\therefore S_1 : x^2 + (y-6)^2 = 8$ (5)

i.e. $x^2 + y^2 - 12y + 28 = 0$.

$S_2 : (x-3)^2 + (y+3)^2 = 2$ (5)

i.e $x^2 + y^2 - 6x + 6y + 16 = 0$

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$AC : CB = 2\sqrt{2} : \sqrt{2} = 2:1$ (5)

$\therefore C \equiv \left(\frac{6+0}{3}, \frac{-6+6}{3}\right) \equiv (2,0)$ (5)

Let m be the slope of the other common tangent through C.

$\therefore \tan\theta = \frac{1}{2} = \left|\frac{m-(-3)}{1+m(-3)}\right|$ (5)

$\Leftrightarrow \quad 1-3m = 2m+6$ or $3m-1 = 2m+6$

$\Leftrightarrow \quad m = -1$ or $m = 7$

$\therefore \quad m = 7.$ (5)

$\therefore$ The required equation is $y - 0 = 7(x-2)$. (5)

i.e. $7x - y - 14 = 0$.

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Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle.

Since S passes through the origin, $c = 0$. (5)

As S bisects the circumference of S_1, the common chord passes through A.

The common chord is $S - S_1 \equiv 2gx + (2f + 12)y - 28 = 0$ (5)

$A \equiv (0,6)$ is on $S - S_1 = 0$, we have

$$(2f + 12)(6) - 28 = 0. \quad (5)$$

$$(f + 6)(3) - 7 = 0, \text{ which gives us } f = -\frac{11}{3}. \quad (5)$$

Since S is orthogonal to S_2, $2g(-3) + 2f(3) = 0 + 16$. (5)

$$\therefore -3g + 3\left(\frac{-11}{3}\right) = 8, \text{ which gives us } \Rightarrow g = -\frac{19}{3}. \quad (5)$$

$\therefore$ The required circle is

$$x^2 + y^2 + 2\left(\frac{-19}{3}\right)x + 2\left(\frac{-11}{3}\right)y = 0 \quad (5)$$

i.e. $3x^2 + 3y^2 - 38x - 22y = 0.$

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17. (*a*) Write down $\cos(A+B)$ and $\cos(A-B)$ in terms of $\cos A$, $\cos B$, $\sin A$ and $\sin B$.

Hence, show that $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

Deduce that $\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$.

Solve the equation $\cos 9x+\cos 7x+\cot x(\cos 9x-\cos 7x)=0$.

(*b*) In the usual notation, state and prove the **Cosine Rule** for a triangle ABC.

Let $x\neq n\pi+\frac{\pi}{2}$ for $n\in\mathbb{Z}$. Show that $\sin 2x=\frac{2\tan x}{1+\tan^2 x}$.

In a triangle ABC, it is given that $AB=20$ cm, $BC=10$ cm and $\sin 2B=\frac{24}{25}$.

Show that there are two distinct such triangles and find the length of AC for each.

(*c*) Solve the equation $\sin^{-1}\left[(1+e^{-2x})^{-\frac{1}{2}}\right]+\tan^{-1}(e^x)=\tan^{-1}(2)$.

(*a*)

$\cos(A+B)=\cos A\cos B-\sin A\sin B$ ⟶ (1) **5**

$\cos(A-B)=\cos A\cos B+\sin A\sin B$ ⟶ (2) **5**

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(1) + (2)

$\cos(A+B)+\cos(A-B)=2\cos A\cos B$ **5**

Taking $A+B=C$ and $A-B=D$, we have $A=\frac{C+D}{2}, B=\frac{C-D}{2}$

$\therefore\quad \cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$. **5**

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Now, $\cos C-\cos D=\cos C+\cos(\pi-D)$ **5**

$=2\cos\left(\frac{C+(\pi-D)}{2}\right)\cos\left(\frac{C-(\pi-D)}{2}\right)$

$=2\sin\left(\frac{D-C}{2}\right)\sin\left(\frac{C+D}{2}\right)$

$=-2\sin\left(\frac{C-D}{2}\right)\sin\left(\frac{C+D}{2}\right)$. **5**

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$\cos 9x + \cos 7x + \cot x\ (\cos 9x - \cos 7x) = 0$ $(\sin x \neq 0)$

$\therefore 2\cos 8x \cos x + \dfrac{\cos x}{\sin x}(-2\sin 8x \sin x) = 0$ (5)

$\therefore \cos x = 0$ or $(\cos 8x - \sin 8x) = 0.$ (5)

$\therefore \cos x = 0$ or $\tan 8x = 1.$

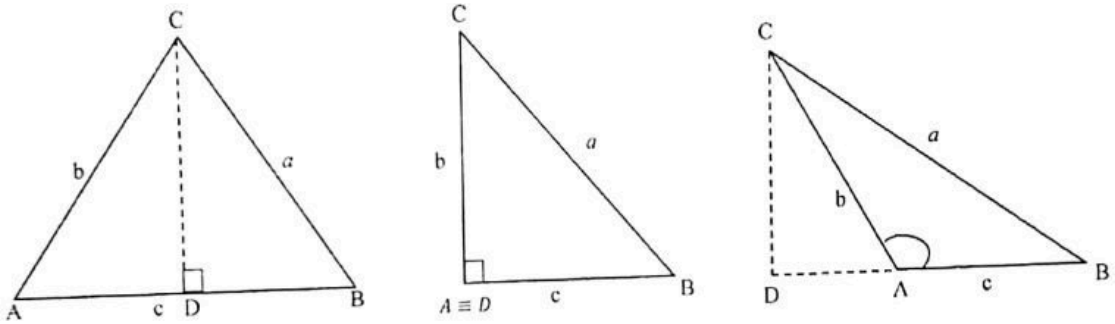
$x = 2m\pi \pm \dfrac{\pi}{2}$ for $m \in \mathbb{Z}$ or $8x = n\pi + \dfrac{\pi}{4}$ for $n \in \mathbb{Z}$

$x = 2m\pi \pm \dfrac{\pi}{2}$ for $m \in \mathbb{Z}$ or $x = \dfrac{n\pi}{8} + \dfrac{\pi}{32}$ for $n \in \mathbb{Z}.$ (5) + (5) **20**

(*b*)

Cosine Rule: Let ABC be a triangle. (5)

Then $a^2 = b^2 + c^2 - 2bc\cos A$



Proof: Let D be the perpendicular of the foot from C on AB. Then by the Pythagoras Theorem.

$BC^2 = BD^2 + DC^2$ ——— (1) (5)

Case (i) A is acute;

$DC = b\sin A$

$DB = c - b\cos A$ (5)

Case (ii) A is obtuse

$DC = b\sin(\pi - A) = b\sin A$

$DB = c + b\cos(\pi - A) = c - b\cos A$ (5)

$\therefore$ In both cases, (1) gives us $a^2 = b^2\sin^2 A + (c - b\cos A)^2$

$= b^2\sin^2 A + c^2 - 2bc\cos A + b^2\cos^2 A$

$= b^2 + c^2 - 2bc\cos A$ $(\because \sin^2 A + \cos^2 A = 1)$ (5)

When $A = \dfrac{\pi}{2}$, $\cos A = 0$ and this holds in that case too. (5)

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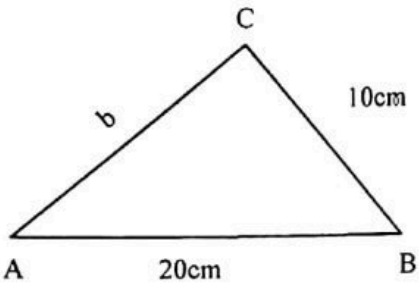
Let $x \neq n\pi + \frac{\pi}{2}$. $(\cos x \neq 0)$

$\sin 2x = \frac{2\sin x \cos x}{\cos^2 x} \times \cos^2 x$ (5)

$= \frac{2\tan x}{\sec^2 x}$

$= \frac{2\tan x}{1+\tan^2 x}$ (5)

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$\sin 2B = \frac{24}{25} \Rightarrow B$ is acute

$\therefore \frac{2t}{1+t^2} = \frac{24}{25}$, where $t = \tan B$ (5)

$12t^2 - 25t + 12 = 0$

$(4t-3)(3t-4) = 0$

$t = \frac{3}{4}$ or $\frac{4}{3}$ (5) + (5)

$\therefore$ two distinct solutions for B.

$\therefore$ two distinct such triangles.

B is an acute angle $\cos B = \frac{3}{5}$ or $\cos B = \frac{4}{5}$

When $\cos B = \frac{3}{5}$; $AC^2 = (20)^2 + 10^2 - 2(20)(10)\left(\frac{3}{5}\right) \Rightarrow AC = 2\sqrt{65}$. (5)

When $\cos B = \frac{4}{5}$; $AC^2 = (20)^2 + (10)^2 - 2(20)(10)\left(\frac{4}{5}\right) \Rightarrow AC = 6\sqrt{5}$. (5)

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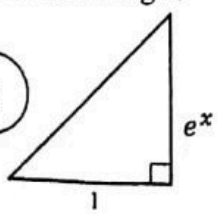
(c)

Let $\alpha = \sin^{-1}(1+e^{-2x})^{-\frac{1}{2}}$. Since $(1+e^{-2x})^{-\frac{1}{2}} > 0$, α is an acute angle.

Then $\sin\alpha = \left(1+e^{-2x}\right)^{-\frac{1}{2}} = \frac{e^x}{\sqrt{1+(e^x)^2}}$

$\therefore \tan\alpha = e^x$. (5)

(5)



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Then, the given equation becomes $\alpha + \alpha = \lambda$.

$\therefore \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \tan\lambda$ **5**

$\Rightarrow \quad \dfrac{2e^x}{1-e^{2x}} = 2$ **5**

$\Rightarrow \quad e^x = 1 - e^{2x}$

$\Rightarrow \quad e^{2x} + e^x - 1 = 0$

$\Rightarrow \quad e^x = \dfrac{-1 \pm \sqrt{5}}{2}$ **5**

Since $e^x > 0$, $(-)$ sign cannot be taken.

$\therefore \quad e^x = \dfrac{-1+\sqrt{5}}{2}$ **5**

$\therefore \quad x = \ln\left(\dfrac{\sqrt{5}-1}{2}\right)$. **5**

Note that this value of x satisfies the given equation.

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