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Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2021 (2022)

10 - Combined Mathematics II

Marking Scheme

This document has been prepared for the use of Marking Examiners. Some changes would be made according to the views presented at the Chief Examiners' meeting.

Amendments to be included

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G. C. E (Advanced Level) Examination - 2021(2022)

10 - Combined Mathematics II

Distribution of Marks

Paper II

Part A	**= 10 × 25**	**= 250**
Part B	**= 05 × 150**	**= 750**

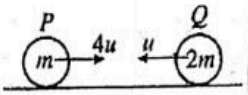
Total $= \frac{1000}{10}$

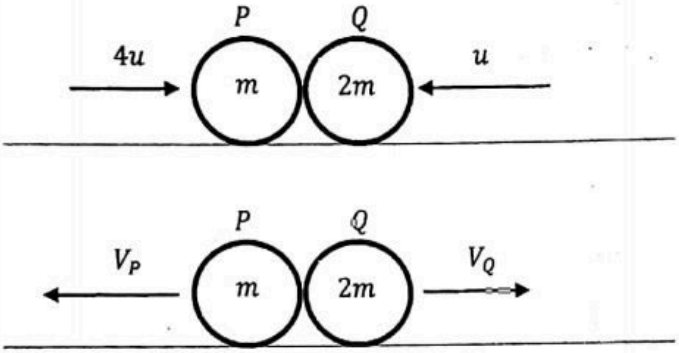
Final marks = 100

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Part A

1. A particle P of mass m and a particle Q of mass $2m$ moving on a smooth horizontal table along the same straight line towards each other with speeds $4u$ and u, respectively, collide directly. The coefficient of restitution between P and Q is $\frac{4}{5}$. Show that the particles P and Q move away from each other after the collision. Find the time taken, after the collision, for P and Q to be at a distance a apart.





$\underline{I} = \Delta(m\underline{V})$ for the system:

$$0 = (2mV_Q - mV_P) - (4mu - 2mu) \quad (5)$$

$$\Rightarrow 2V_Q - V_P = 2u \quad (1)$$

$I = \Delta(m\underline{v})$ for P and Q

By Newton's Experimental Law,

$$V_Q + V_P = \tfrac{4}{5}(4u + u) \quad (5)$$

Newton's Experimental Law

$$V_Q + V_P = 4u \quad (2)$$

$$\therefore V_Q = 2u \text{ and } V_P = 2u. \quad (5)$$

For both V_p and V_Q

(1) + (2): $V_Q > 0$ and $V_P > 0$.

$\therefore P$ and Q move away from each other after the collision. (5)

For both $V_p > 0$ and $V_Q > 0$ or equivalent

$$\underline{V}(P,Q) = \underline{V}(P,E) + \underline{V}(E,Q)$$

$$= \overleftarrow{2u} + \overleftarrow{2u}$$

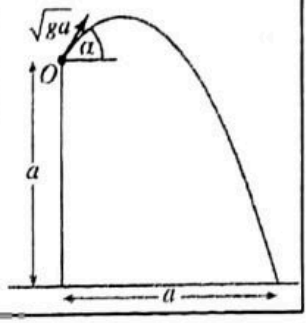
$$= 4u \leftarrow$$

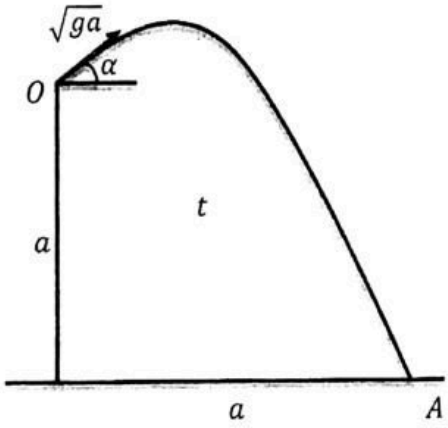
The required time $= \frac{a}{4u}$. (5)

$\frac{a}{4u}$ seen.

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2. A particle is projected from a point O at a vertical distance a above a horizontal ground with initial velocity $\sqrt{ga}$ and at an angle $\alpha\left(0 < \alpha < \frac{\pi}{2}\right)$ to the horizontal, as shown in the figure. The particle strikes the ground at a horizontal distance a from O. Show that $\tan\alpha = 1 + \sqrt{2}$.





From O to A $S = ut + \frac{1}{2}at^2$:

$\rightarrow \quad a = \sqrt{ga}\cos\alpha\ t$ ---------- (1) (5) $\rightarrow s = u + \frac{1}{2}at^2$

$\uparrow \quad -a = \sqrt{ga}\sin\alpha\ t - \frac{1}{2}gt^2$ ---------- (2) (5) $\uparrow s = u + \frac{1}{2}at^2$

(1) given us $t = \frac{a}{\sqrt{ga}\cos\alpha}$.

(2) given us $-a = a\tan\alpha - \frac{1}{2}g\frac{a^2}{ga\cos^2(\alpha)}$.

$\therefore \quad -2 = 2\tan\alpha - (1 + tan^2\alpha).$ (5) For the quadratic in tan α

i.e. $tan^2\alpha - 2\tan\alpha - 1 = 0.$

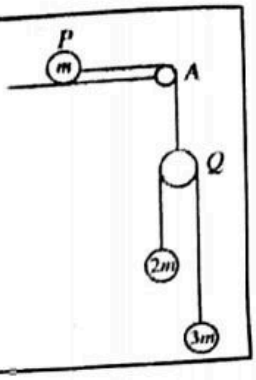
$\therefore \tan\alpha = \frac{-2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ (5) both $\pm$

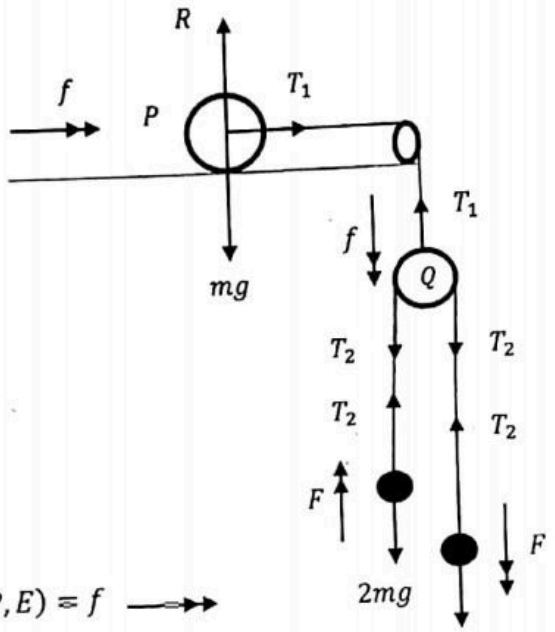
$(-)$ sign is not possible as $0 < \alpha < \frac{\pi}{2}$.

$\therefore \tan\alpha = 1 + \sqrt{2}.$ (5) Selecting the correct sign

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3. A particle P of mass m is placed on a smooth horizontal table and is connected to a light smooth pulley Q by a light inextensible string which passes over a fixed small smooth pulley at the point A of the edge of the table. A light inextensible string which passes over the pulley Q is connected to particles of masses $2m$ and $3m$, as shown in the figure. The particles and the strings lie in a vertical plane. The system is released from rest with the strings taut. Obtain equations sufficient to determine the acceleration of Q.





5

$f \rightarrow$ $\downarrow f$

$\underline{a}(P, E) = f \longrightarrow$

$\underline{a}(3m, Q) = F \downarrow$

Applying $\underline{F} = m\underline{a}$:

$\underline{F} = m\underline{a}$ for P

(P) $\longrightarrow$ $T_1 = mf$ (5)

$\underline{F} = m\underline{a}$ for Q

(Q) $\downarrow$ $2T_2 - T_1 = 0$ (5)

$\underline{F} = m\underline{a}$ for $2m$

(2m) $\uparrow$ $T_2 - 2mg = 2m(F - f)$ (5)

$\underline{F} = m\underline{a}$ for $3m$

(3m) $\downarrow$ $3mg - T_2 = 3m(F + f)$ (5)

or

(Q) (2m) and (3m) $\downarrow$ $T_1 - 2mg - 3mg = 2m(f - F) - 3m(f + F)$

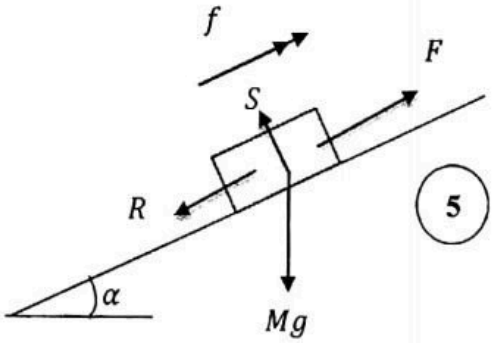
$\underline{F} = m\underline{a}$ for Q, $2m$ and $3m$

Note: Any 4 correct independent equations (20)

25

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4. A car of mass M kg moves upwards with a constant acceleration along a straight road of inclination $\sin^{-1}\left(\frac{1}{20}\right)$ to the horizontal. There is a constant resistance of R N to its motion. The distance travelled by the car to increase its speed from 36 km h^{-1} to 72 km h^{-1} is 500 m. Obtain equations sufficient to determine the power exerted by the car when its speed is 54 km h^{-1}.



5

For forces with or without S

$\sin\alpha = \frac{1}{20}$

$\frac{36 \times 1000}{3600} = 10\ ms^{-1}$

$\frac{72 \times 1000}{3600} = 20\ ms^{-1}$ (5)

$\frac{54 \times 1000}{3600} = 15\ ms^{-1}$

For all three conversions of speeds

↗ $v^2 = u^2 + 2as$:

$20^2 = 10^2 + 2f(500)$ (5)

$f = \frac{150}{500} = \frac{3}{10} ms^{-2}$

Applying $v^2 = u^2 + 2as$ ↗

↗ $\underline{F} = m\underline{a}$:

$F - R - Mg\sin\alpha = Mf$ (5)

Applying $\underline{F} = m\underline{a}$ ↗

$P = F \cdot V$

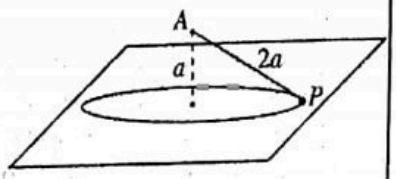
$= F \cdot 15$ (5)

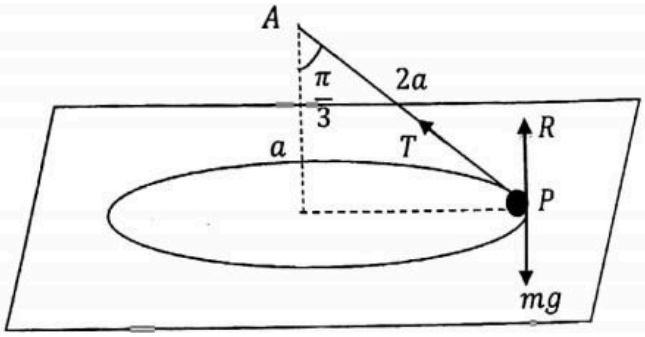
$P = F \cdot 15$ seen

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5. One end of a light inextensible string of length $2a$ is attached to a fixed point A which is at a distance a vertically above a smooth horizontal table. A particle P of mass m, attached to the other end of the string, moves in a horizontal circle on the table with the string taut and with uniform speed $\sqrt{\frac{ga}{2}}$ (see the figure). Show that the magnitude of the normal reaction on the particle P from the table is $\frac{5}{6}mg$





5 For the forces

Applying $\underline{F} = m\underline{a}$:

$\leftarrow T\sin\frac{\pi}{3} = m.\frac{ga}{2(2a\sin\frac{\pi}{3})}$ 5 $\underline{F} = m\underline{a}$ ←

$\therefore T = \frac{mg}{3}.$ 5 $T = \frac{mg}{3}$ seen or implied

$\uparrow R - mg + T\cos\frac{\pi}{3} = 0$ 5 $\underline{F} = m\underline{a}$ ↑

$\therefore R = mg - \frac{mg}{6}$

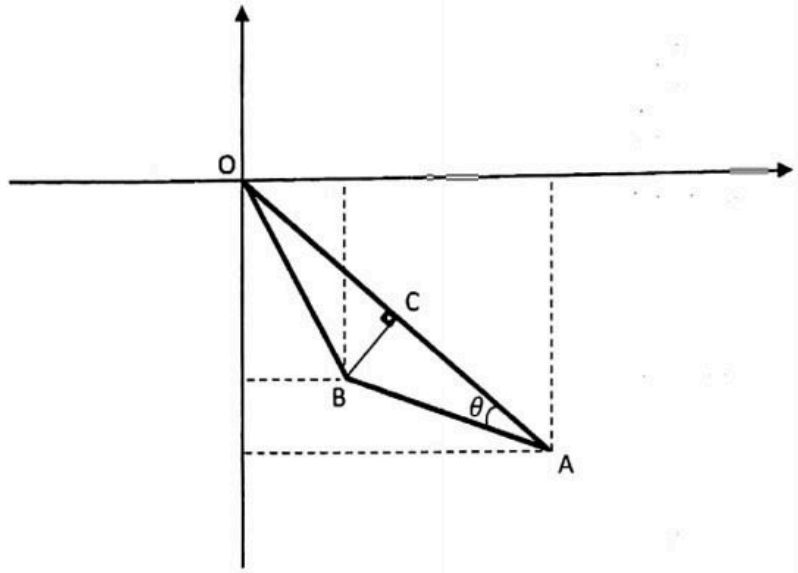
$= \frac{5}{6}mg$ 5 Work leading to the answer

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6. In the usual notation, the position vectors of two points A and B, with respect to a fixed origin O are $2\mathbf{i}-3\mathbf{j}$ and $\mathbf{i}-2\mathbf{j}$, respectively. Using $\overrightarrow{AO}\cdot\overrightarrow{AB}$, find $O\hat{A}B$.

Let C be the point on OA such that $O\hat{C}B=\frac{\pi}{2}$. Find $\overrightarrow{OC}$.



$\overrightarrow{OA} = 2\underline{i} - 3\underline{j}$ and $\overrightarrow{OB} = \underline{i} - 2\underline{j}$

$\therefore \overrightarrow{AO} = -2\underline{i} + 3\underline{j}$ and

$\overrightarrow{AB} = \left(\underline{i} - 2\underline{j}\right) - \left(2\underline{i} - 3\underline{j}\right)$ (5)

$= -\underline{i} + \underline{j}$

$\overrightarrow{AB}$ is terms of $\underline{i}$ and $\underline{j}$

$\overrightarrow{AO}\cdot\overrightarrow{AB} = |\overrightarrow{OA}||\overrightarrow{AB}|\cos\theta$ (5)

$2 + 3 = \sqrt{13}\sqrt{2}\cos\theta$

Definition of $\overrightarrow{AO}\cdot\overrightarrow{AB}$ or equivalent

$\therefore \quad \cos\theta = \frac{5}{\sqrt{26}}$ (5)

$\cos\theta = \frac{5}{\sqrt{26}}$ seen

$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{\sqrt{26}}\right)$

$\overrightarrow{OC} = \lambda\overrightarrow{OA}$, where $\lambda \in \mathbb{R}$, and $\overrightarrow{CB} = \left(\underline{i} - 2\underline{j}\right) - \lambda(2\underline{i} - 3\underline{j})$

$\overrightarrow{OA}\cdot\overrightarrow{CB} = 0$ gives as $2(1-2\lambda) - 3(-2+3\lambda) = 0$ (5)

condition for $\hat{OCB} = \frac{\pi}{2}$ using dot product

$\therefore \quad \lambda = \frac{8}{13}$ (5)

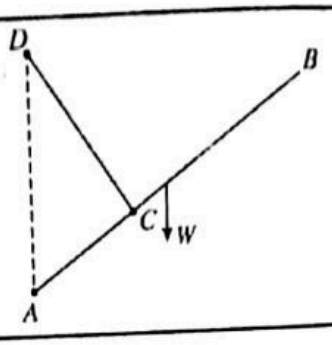
value of λ or equivalent

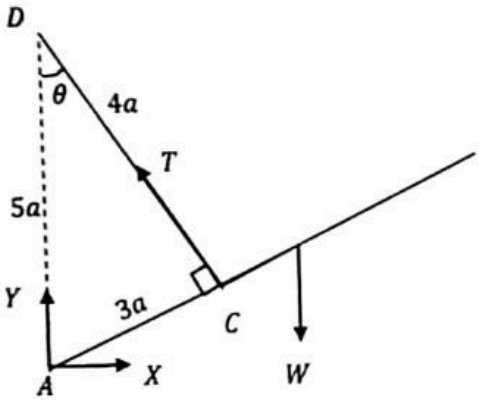
and $\overrightarrow{OC} = \frac{8}{13}(2\underline{i} - 3\underline{j})$

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7. A uniform rod AB of length $8a$ and weight W has its end A smoothly hinged to a fixed point. One end of a light inextensible string of length $4a$ is attached to the point C on the rod such that $AC = 3a$, and the other end is attached to a fixed point D vertically above A such that $AD = 5a$ (see the figure). The rod is in equilibrium. Show that the tension of the string is $\frac{16}{15}W$. Also, find the horizontal component of the reaction at A.





5 For the forces

$\hat{ACD} = \frac{\pi}{2}$ 5

$\hat{ACD} = \frac{\pi}{2}$ seen

For the equilibrium of the rod:

$\curvearrowleft_A \quad W \times 4a\cos\theta - T \times 3a = 0$ 5

An equation sufficient to find T.

$$\therefore T = \frac{4W}{3}\cos\theta$$

$$= \frac{4W}{3} \times \frac{4}{5} = \frac{16W}{15}$$ 5

work leading to the answer

$$\rightarrow X = T\sin\theta$$

$$= \frac{16W}{15} \times \frac{3}{5}$$

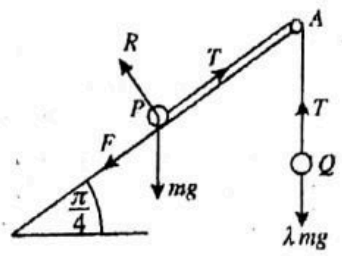
$$= \frac{16W}{25}$$ 5

$\frac{16W}{25}$ seen.

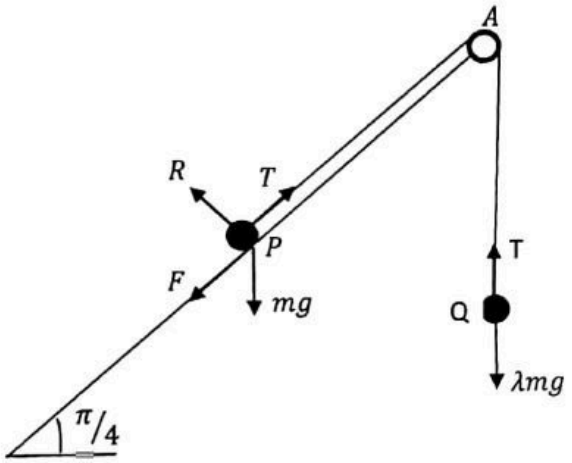
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8. A particle P of mass m is placed on a rough plane inclined at an angle $\frac{\pi}{4}$ to the horizontal. One end of a light inextensible string which passes over a fixed small smooth pulley fixed to the edge of the inclined plane at A, is attached to the particle P and the other end to a particle Q of mass λmg, as shown in the figure. The coefficient of friction between the particle P and the inclined plane is $\frac{1}{2}$. The line PA is a line of greatest slope of the inclined plane and the particles P and Q stay in equilibrium with the string taut.



Show that $\frac{1}{2\sqrt{2}} \leq \lambda \leq \frac{3}{2\sqrt{2}}$. (The relevant forces are marked in the figure.)



For the equilibrium:

(P) 

$$R - mg\cos\left(\frac{\pi}{4}\right) = 0 \quad (5)$$

$$\therefore \quad R = \frac{mg}{\sqrt{2}}$$

An equation leading to the value of R.

(Q) 

$$T - \lambda mg = 0 \quad (5)$$

$$\therefore \quad T = \lambda mg$$

An equation leading to the value of T.

(P) 

$$T - F - mg\sin\left(\frac{\pi}{4}\right) = 0 \quad (5)$$

$$\therefore \quad F = \lambda mg - \frac{mg}{\sqrt{2}} = \frac{mg}{\sqrt{2}}(\sqrt{2}\lambda - 1)$$

An equation leading to the value of F.

For the equilibrium of P:

$$\frac{1}{2} \geq \frac{|F|}{R} \quad (5)$$

Condition for non-slipping with absolute value.

$$\therefore \quad |\sqrt{2}\lambda - 1| \leq \frac{1}{2}$$

work leading to the answer

$$(5) \quad \therefore \quad \frac{1}{2\sqrt{2}} \leq \lambda \leq \frac{3}{2\sqrt{2}}$$

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9. Let A and B be two **independent** events of a sample space Ω. In the usual notation, it is given that $P(A)=\frac{1}{5}$ and $P(B)=\frac{3}{4}$. Find $P(A \cup B)$, $P(A \mid A \cup B)$ and $P(B \mid A')$, where A' denotes complementary event of A.

$P(A)=\frac{1}{5}, P(B)=\frac{3}{4}$

Since A and B are independent,

For the condition for independence

$P(A \cap B) = P(A) \cdot P(B)$ **5**

$= \frac{3}{20}$

Formula a for $P(A \cup B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ **5**

$= \frac{1}{5} + \frac{3}{4} - \frac{3}{20} = \frac{4}{5}$

$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{1/5}{4/5} = \frac{1}{4}$ **5**

$\frac{1}{4}$ or equivalent seen.

$P(B|A') = \frac{P(B \cap A')}{P(A')}$

$P(B \cap A') = P(B) - P(A \cap B) = \frac{3}{4} - \frac{3}{20} = \frac{3}{5}$ **5**

$\frac{3}{5}$ or equivalent seen.

(OR $P(B \cap A') = P(B) \cdot P(A') = \frac{3}{4} \times \frac{4}{5} = \frac{3}{5}$)

$P(B|A') = \frac{3/5}{4/5} = \frac{3}{4}$ **5**

$\frac{3}{4}$ or equivalent seen.

25

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10. A set of five observations of positive integers has mean 6 and range 10. It has two modes. If the median is different from the modes, find the five observations.

Let the numbers in the increasing order be

a, a, b, c, c

Since the range is 10, we have $c - a = 10$. (5) Condition for the range

$\therefore c = a + 10$ ———(1)

Since the mean is 6, we have $\frac{2a+b+2c}{5} = 6$. ———(2) (5) For this or an equivalence

(1) and (2)gives us $4a + b + 20 = 30$

i.e. $4a + b = 10$ ———(3) (5) An equation sufficient to determine the observations

Since a and b are positive integers,

Then, (3) Implies that $4a \leq 9$ and the only possible values for a are 1 and 2.

If $a = 1$, then $b = 6$.

If $a = 2$, then $b = 2$, and it is not possible as the median is different from the modes. (5) Mean ≠ mode used.

$\therefore$ The numbers are $1, 1, 6, 11, 11$. (5) 1, 1, 6, 11, 11 seen.

25

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11. (*a*) A particle P, projected with a velocity u m s^{-1} vertically upwards from a point O, reaches a point A after 4 seconds and comes back to A again after another 2 seconds. At the instant when the particle P is at A for the second time, another particle Q is projected with the same velocity u m s^{-1} vertically upwards from O. Sketch the velocity-time graph for the motions of P and Q, in the same diagram.

Hence, find the value of u and the height of OA in terms of g, and the time taken by Q to collide with P.

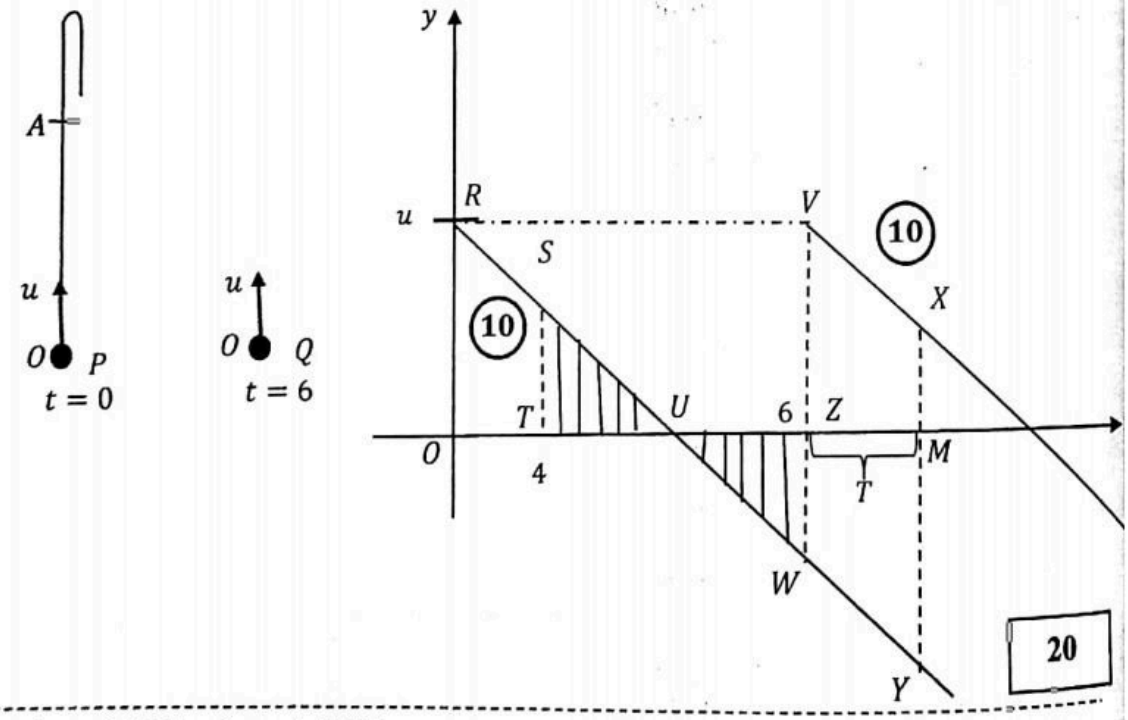
(*b*) A ship S is sailing due north with uniform speed u km h^{-1} relative to earth. At a certain instant, a boat P is at a distance d km east of S and another boat Q is at a distance $\sqrt{3}\,d$ km south of S. The boat P travels in a straight line path intending to intercept S with uniform speed $2u$ km h^{-1} relative to earth and the boat Q travels in a straight line path intending to intercept P with uniform speed $3u$ km h^{-1} relative to earth.

Show that

(i) the time taken by the boat P to intercept the ship S is $\frac{d}{\sqrt{3}u}$ h,

(ii) the boat P intercepts the ship S before the boat Q intercepts the boat P.

(a)



20

Since Area $\Delta STU \equiv$ Area ΔUZW,

we have $TU = UZ$.

$TZ = 2 \Rightarrow TU = 1$. (5)

$\therefore OU = 5$. (5)

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From $\Delta\, ROU$, we have $g = \frac{u}{5}$.

$\therefore u = 5g.$ (5)

From $\Delta\, STU$, we have $g = \frac{ST}{1} = ST.$ (5)

The height of OA = Area of $ORST$.

$= \frac{1}{2}(OR + ST) \times OT$ (5)

$= \frac{1}{2}(u + g) \times 4$ (5)

$= \frac{1}{2} \times 6g \times 4$

$= 12g$ (5)

Let T be the time taken by Q to collide with P.

OA = Area $VZMX$ + Area $WZMY$

= Area $VWYX$ (10)

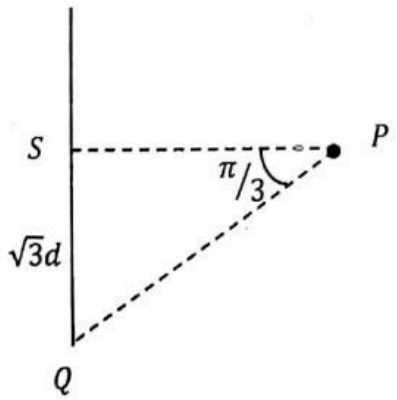
$= \frac{1}{2}(VW + XT) \times ZM$

$\therefore\ 12g = \frac{1}{2}(6g + 6g) \times T$ (10)

$\therefore T = 2$ sec. (5)

60

(b)



$\underline{V}(S, E) = \uparrow u$

$\underline{V}(P, E) = 2u$

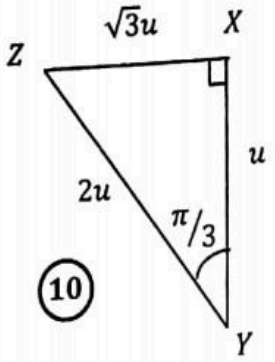
$\underline{V}(Q, E) = 3u$

$\underline{V}(P, S) = \leftarrow$

$\underline{V}(Q, P) =$ 

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(i)



10

$\underline{V}(P,S) = \underline{V}(P,E) + \underline{V}(E,S)$ (5) + (5)

$= \underline{V}(E,S) + \underline{V}(P,E)$

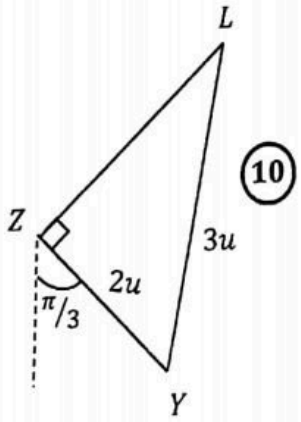
$= \overrightarrow{XY} + \overrightarrow{YZ}$

$= \overrightarrow{XZ}$

The required time $= \frac{d}{XZ} = \frac{d}{\sqrt{3}u}\ h.$ (5)

25

(ii)



10

$\underline{V}(Q,P) = \underline{V}(Q,E) + \underline{V}(E,P)$ (5)

$= \underline{V}(E,P) + \underline{V}(Q,E)$

$= \overrightarrow{ZY} + \overrightarrow{YL}$

$= \overrightarrow{ZL}$

$ZL = \sqrt{(3u)^2 - (2u)^2} = \sqrt{5}u$ (5)

Let t_2 be the time taken by Q to intercept P.

Then, $t_2 = \frac{\sqrt{3}\, d \sec(\pi/6)}{\sqrt{5}\, u}$

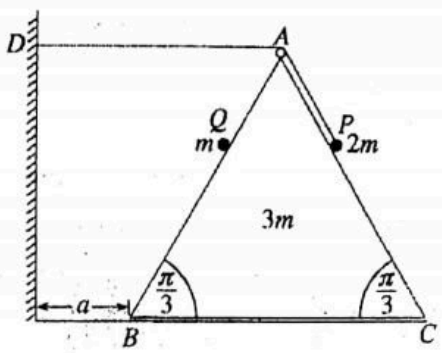
$= \frac{2d}{\sqrt{5}\, u}\ h.$ (10)

$\therefore\ t_1 < t_2.$ (10)

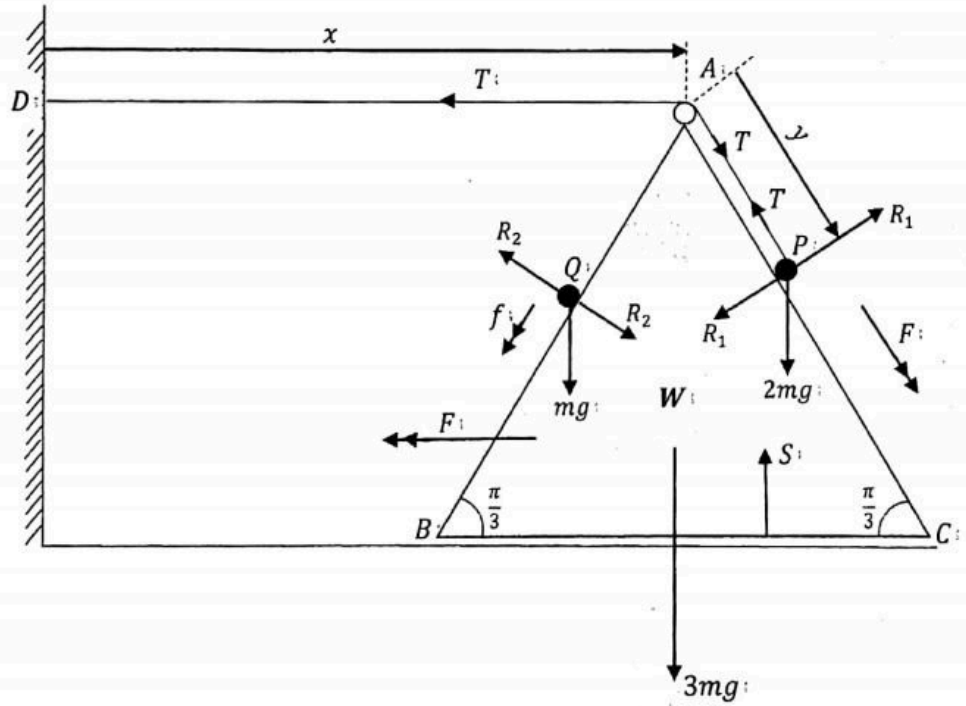
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12. (a) Equilateral triangle ABC in the figure is the vertical cross-section through the centre of gravity of a smooth uniform wedge of mass $3m$ with $AB = BC = AC = 6a$ such that the face containing BC is placed on a smooth horizontal floor. The lines AB and AC are lines of greatest slope of the faces containing those. The point D is a fixed point on the vertical wall which is at a distance a from the point B of the wedge, and in the plane of ABC such that AD is horizontal. One end of a light inextensible string of length $5a$ passing over a small smooth pulley fixed at A is attached to a particle P of mass $2m$ kept on AC and the other end is attached to the fixed point D on the wall. A particle Q of mass m is held on AB. The system is released from the rest with $AP = AQ = a$, as shown in the figure. Obtain equations sufficient to determine the velocity of Q relative to the wedge at the instant when the wedge strikes the wall.



a)



$x + y = constant.$ (5)

$\therefore \ddot{x} + \ddot{y} = 0$ _____(1) (5)

Let $\underline{a}(W, E) = F \leftarrow$

$\therefore \ \underline{a}(P, W) = F \searrow$ (by (1)) (5)

Also, let $\underline{a}(Q, W) = f \swarrow$

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Applying $\underline{F} = m\underline{a}$: (5) (for forces) (5) (for acceleration)

(P) ↘ $2mg\sin\left(\frac{\pi}{3}\right) - T = 2m\left(F - F\cos\left(\frac{\pi}{3}\right)\right)$ (5) (for equation)

(Q) ↙ $mg\sin\left(\frac{\pi}{3}\right) = m\left(f + F\cos\left(\frac{\pi}{3}\right)\right)$ (5) (for equation)

(5) (5)

For the system $(P, Q, and\ W)$, ←

$T = 3mF + 2m\left(F - F\cos\left(\frac{\pi}{3}\right)\right) + m\left(F + f\cos\left(\frac{\pi}{3}\right)\right)$ (5) (for equation)

(5) (5) (5) (5)

Applying $S = ut + \frac{1}{2}at^2$:

← (W) $a = \frac{1}{2}ft^2$ (5)

Applying $v = u + at$:

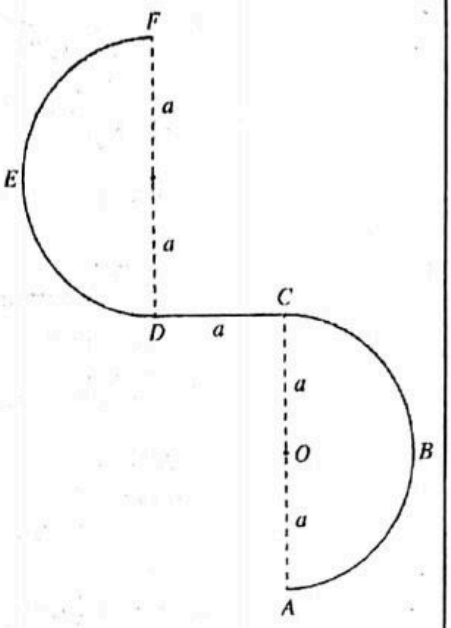
$v = Ft$ (5)

80

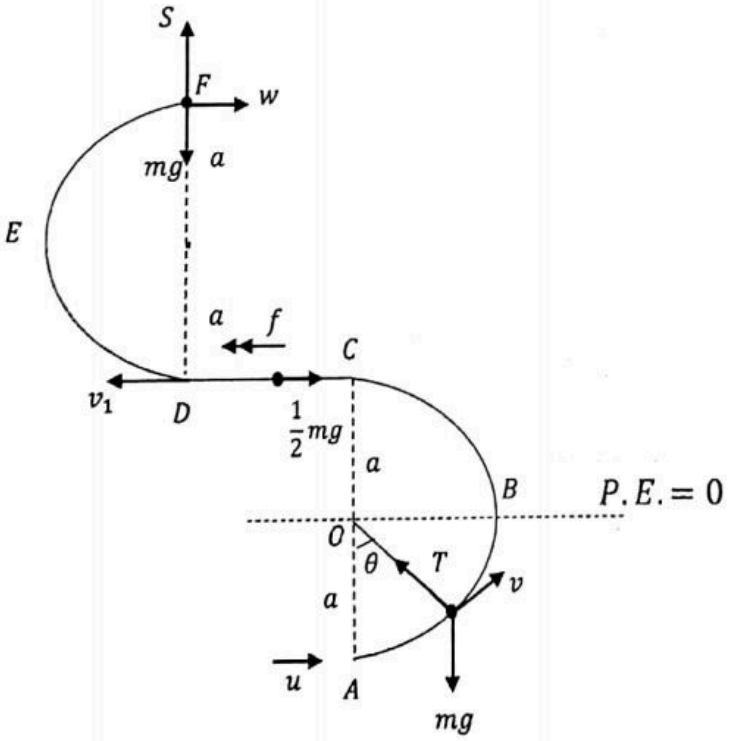
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(*b*) A thin wire *ABCDEF* is fixed in a vertical plane, as shown in the figure. The portion *ABC* is a thin **smooth** semicircular wire with centre *O* and radius *a*. The portion *CD* is a thin **rough** horizontal wire of length *a*. The portion *DEF* is also a thin **smooth** semicircular wire of radius *a*. The diameters *AC* and *DF* are vertical. A small smooth bead *P* of mass *m* is placed at *A* and is given a velocity $u\left(>3\sqrt{ag}\right)$ horizontally, and begins to move along the wire. It is given that the magnitude of the frictional force on the bead from the wire, during its motion from *C* to *D*, is $\frac{1}{2}mg$. Show that the speed v of the bead *P*, during its motion from *A* to *C*, when $\overrightarrow{OP}$ makes an angle $\theta\ (0 \le \theta \le \pi)$ with $\overrightarrow{OA}$, is given by $v^2 = u^2 - 2ag(1 - \cos\theta)$.

Show that the speed w of the bead *P* just before it leaves the wire at *F* is given by $w^2 = u^2 - 9ag$, and find the reaction on the bead *P* from the wire at that instant.



b)



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By the conservation of energy,

$\frac{1}{2}mv^2 - mga\cos\theta = \frac{1}{2}mu^2 - mga$ (15) PE (5)+KE (5)+Equation (5)

$\therefore\ v^2 = u^2 - 2ga(1-\cos\theta)$ (5)

When $\theta = \pi$, $v^2 = u^2 - 4ga$ _____(1) (5)

25

From C to D, $\leftarrow$ $\underline{F} = m\underline{a}$:

$-\frac{1}{2}mg = mf$ (5)

$\therefore\quad f = -\frac{g}{2}$ (5)

$\leftarrow\ v^2 = u^2 + 2as$: ${v_1}^2 = (u^2 - 4ga) - 2\cdot\frac{g}{2}a$

$= u^2 - 5ga.$ (10)

Using (1), we have $w^2 = {v_1}^2 - 4ga$ (10)

$= u^2 - 9ga.$ (5)

$\underline{F} = m\underline{a}$ $\downarrow$ at F:

$mg - S = m\frac{w^2}{a}$ (5)

$\therefore\ S = mg - \frac{m}{a}(u^2 - 9ga)$

$= \frac{m}{a}(10ag - u^2)$ (5)

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13. One end of a light elastic string of natural length $4a$ is attached to a fixed point O and the other end to a particle P of mass m. The particle hangs in equilibrium at a distance $5a$ below O.

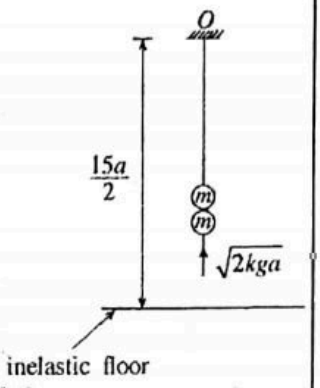
Show that the modulus of elasticity of the string is $4mg$.

Now, another particle Q of mass m moving vertically upwards collides and coalesces with P, and form a combined particle R. The speed of the particle Q just before it collides with the particle P is $\sqrt{2kga}$. Find the velocity with which R begins to move.

Show that, in the subsequent motion while the string is not slack, the distance x from O to the combined particle R satisfies the equation $\ddot{x}+\frac{g}{2a}(x-6a)=0$.

By writing $X = x-6a$, show that, $\ddot{X} + w^2X = 0$ where $w = \sqrt{\frac{g}{2a}}$.

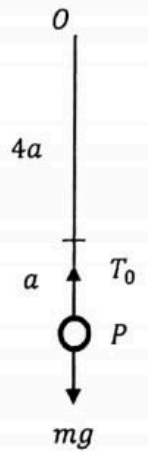
Find the centre of the above simple harmonic motion and using the formula $\dot{X}^2 = w^2(c^2 - X^2)$, find the amplitude c.



Show that if $k > 3$, then the string becomes slack,

Now, let $k = 8$. Find the time taken by the combined particle R to strike an **inelastic horizontal floor** at a distance $\frac{15}{2}a$ below the point O, from the instant of coalescing of the particles P and Q.

Also, find the maximum height reached by the combined particle R after striking the floor.



For the equilibrium of P,

$\uparrow T_0 = mg$ (5)

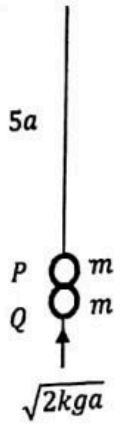
$T_0 = \frac{\lambda a}{4a} = \frac{\lambda}{4}$ (5)

$\therefore \lambda = 4mg$

(5)

15

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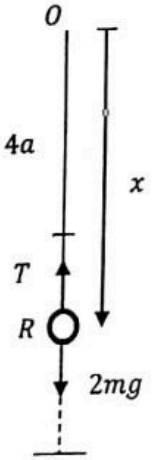


Applying $\underline{I} = \Delta(m\underline{v})$ for P and Q:

$\uparrow 0 = 2mu - m\sqrt{2kga}$ (5)

$\therefore u = \sqrt{\frac{kga}{2}}$ (5)

10



Applying $\underline{F} = m\underline{a}$ for R:

$\uparrow T - 2mg = -2m\ddot{x}$ (10)

$T = 4mg\frac{(x-4a)}{4a}$ (5)

$\therefore \frac{mg}{a}(x-4a) - 2mg = -2m\ddot{x}$

$\ddot{x} + \frac{g}{2a}(x-6a) = 0$ _____(1) (5)

20

$X = x - 6a$

$\therefore \dot{X} = \dot{x}$

$\therefore \ddot{X} = \ddot{x}$ (5)

Then (1) $\Rightarrow \ddot{X} + \omega^2 X = 0$, where $\omega = \sqrt{\frac{g}{2a}}$. (5)

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Centre is given by $X = 0$.

i.e. $x = 6a$. (5)

$\dot{X}^2 = \omega^2(c^2 - X^2)$_____(2)

When $x = 5a$, we have $X = -a$ and $\dot{X} = -\frac{1}{2}\sqrt{2kga}$. (5)

Then (2) $\Rightarrow \frac{kga}{2} = \frac{g}{2a}(c^2 - a^2)$.

$\Rightarrow ka^2 = c^2 - a^2$.

$\Rightarrow c = \sqrt{k+1}\, a$. (5)

15

Let $k > 3$. Then, $c > 2a$.

$\therefore$ Amplitude $> 2a$. (5)

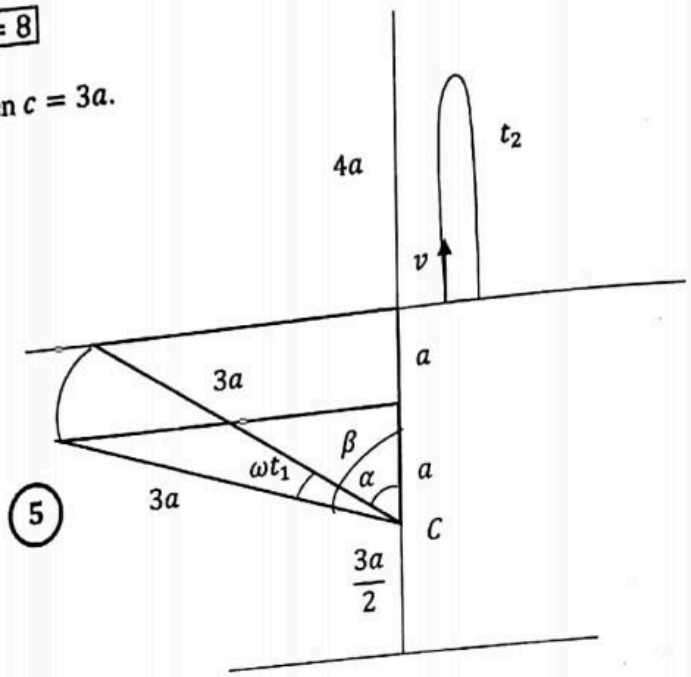
$\therefore$ the string becomes slack. (5)

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$\boxed{k = 8}$

Then $c = 3a$.



(5)

$\cos\beta = \frac{1}{3}$ (5)

$\cos\alpha = \frac{2}{3}$ (5)

$\omega t_1 = \beta - \alpha$

$\therefore\ t_1 = \frac{1}{\omega}(\beta - \alpha)$ (5)

Now, $v^2 = \frac{g}{2a}(9a^2 - 4a^2)$

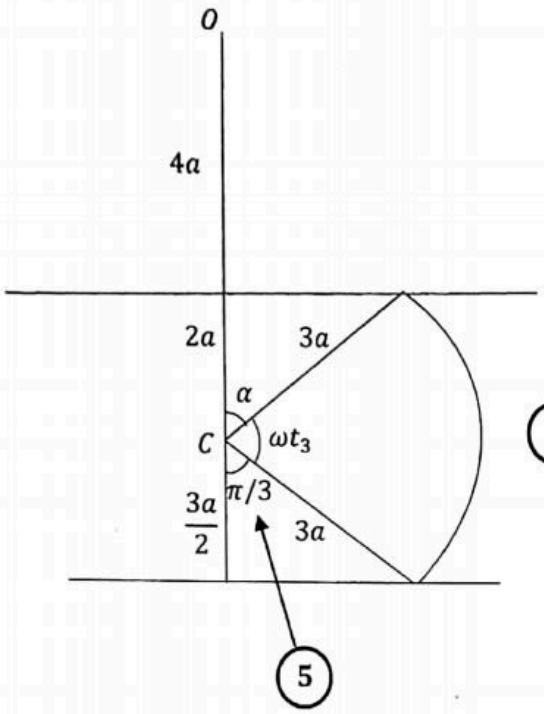
$\therefore v = \sqrt{\frac{5}{2}ga}$ (5)

Under gravity: $\uparrow\ s = ut + \frac{1}{2}at^2$ (5)

$0 = vt_2 - \frac{1}{2}g{t_2}^2.$

$\therefore\ t_2 = \frac{2v}{g} = \frac{2}{g}\sqrt{\frac{5}{2}ga} = \sqrt{\frac{10a}{g}}$ (5)

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5 5

$\omega t_3 = \frac{2\pi}{3} - \alpha$ (5)

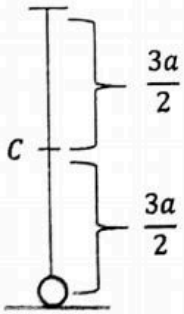
$\therefore\ t_3 = \frac{1}{\omega}\left(\frac{2\pi}{3} - \alpha\right)$

∴ The required time $= t_1 + t_2 + t_3$

$$= \frac{1}{\omega}(\beta - \alpha) + \sqrt{\frac{10a}{g}} + \frac{1}{\omega}\left(\frac{2\pi}{3} - \alpha\right)$$

$$= \sqrt{\frac{10a}{g}} + \sqrt{\frac{2a}{g}}\left\{\frac{2\pi}{3} + \cos^{-1}\left(\frac{1}{3}\right) - 2\cos^{-1}\left(\frac{2}{3}\right)\right\}$$ (10)

60



After hitting the floor, R performs only simple harmonic motion. (5)

∴ The maximum height $= \frac{3a}{2} + \frac{3a}{2}$

$= 3a$ (5)

10

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14. (a) Let **a** and **b** be non-zero and non-parallel vectors, and $\lambda, \mu \in \mathbb{R}$.

Show that if $\lambda\mathbf{a} + \mu\mathbf{b} = \mathbf{0}$, then $\lambda = 0$ and $\mu = 0$.

Let ABC be a triangle. The mid-point of AB is D and the mid-point of CD is E. The lines AE (extended) and BC meet at F. Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$. Using the triangle law of addition, show that $\overrightarrow{AE} = \frac{\mathbf{a}+2\mathbf{b}}{4}$.

Explain why $\overrightarrow{AF} = \alpha\overrightarrow{AE}$ and $\overrightarrow{CF} = \beta\overrightarrow{CB}$, where $\alpha, \beta \in \mathbb{R}$.

Considering the triangle ACF, show that $(\alpha - 4\beta)\mathbf{a} + 2(\alpha + 2\beta - 2)\mathbf{b} = \mathbf{0}$.

Hence, find the values of α and β.

(b) Let ABC be an equilateral triangle of sides $2a$ and let D, E, F be the mid points of AB, BC and AC respectively. Forces of magnitudes $2P$, $\sqrt{3}P$, $2\sqrt{3}P$ and αP act respectively along $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{DC}$ and $\overrightarrow{BC}$. It is given that the resultant of this system of forces is acting parallel to $\overrightarrow{AC}$. Find the value of α.

The system of forces is equivalent to a single force of magnitude R acting through A together with a couple of magnitude G. Find the values of R and G.

Write down the magnitude and the direction of the resultant of this system of forces and find the distance from A to the point at which the line of action of the resultant meets AB.

A couple of magnitude H is now added to the system. The resultant of this new system acts through the point B. Find the value of H and the sense of this couple.

(a)

$\underline{a}, \underline{b} \neq \underline{0}$ and $\underline{a} \nparallel \underline{b}$

$\lambda\underline{a} + \mu\underline{b} = \underline{0}$ _____(1)

If $\lambda \neq 0$, then $\underline{a} = -\frac{\mu}{\lambda}\underline{b}$. (5)

This contradicts the given condition.

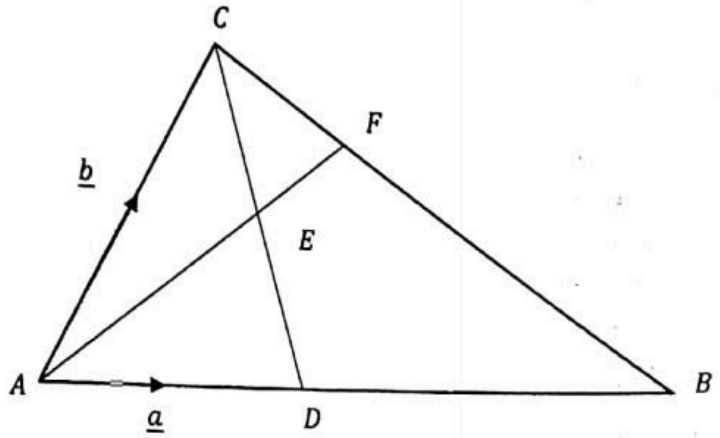
$\therefore \lambda = 0$. (5)

Now, (1) gives us $\mu\underline{b} = \underline{0}$

Since $\underline{b} \neq \underline{0}$, we have $\mu = 0$ (5)

$\therefore \lambda = 0$ and $\mu = 0$

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$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$ (5)

$= \overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC}$ (5)

$= \frac{1}{2}\underline{a} + \frac{1}{2}(\overrightarrow{DA} + \overrightarrow{AC})$ (5)

$= \frac{1}{2}\underline{a} + \frac{1}{2}\left(-\frac{1}{2}\underline{a} + \underline{b}\right)$

$= \frac{\underline{a} + 2\underline{b}}{4}.$ (5)

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$AF \parallel AE$ (or A, E, F are colinear) (5)

$CF \parallel CB$ (or C, F, B are colinear) (5)

10

$\overrightarrow{AF} = \overrightarrow{AC} + \overrightarrow{CF}$ (5)

$\therefore \alpha\overrightarrow{AE} = \underline{b} + \beta\overrightarrow{CB}$

$\therefore \alpha\left(\frac{\underline{a} + 2\underline{b}}{4}\right) = \underline{b} + \beta\left(\overrightarrow{CA} + \overrightarrow{AB}\right)$ (5)

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$\therefore \alpha\underline{a} + 2\alpha\underline{b} = 4\underline{b} + 4\beta(-\underline{b} + \underline{a})$

$\therefore (\alpha - 4\beta)\underline{a} + (2\alpha + 4\beta - 4)\underline{b} = \underline{0}$ (5)

$\underline{a}, \underline{b} \neq 0$ and $\underline{a} \nparallel \underline{b}$ give us,

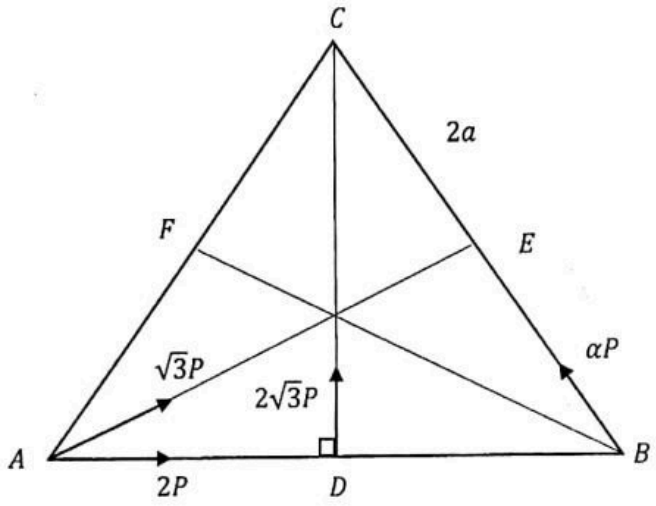
$\alpha - 4\beta = 0$ or $2\alpha + 4\beta - 4 = 0$

$\therefore \alpha = \frac{4}{3}$ and $\beta = \frac{1}{3}$

(5) (5)

25

(b)



$\rightarrow X = 2P + \sqrt{3}P\cos\frac{\pi}{6} - \alpha P\cos\frac{\pi}{3}$ (5)

$= 2P + \frac{3P}{2} - \frac{\alpha P}{2}$

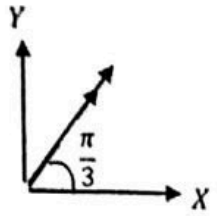
$= \frac{1}{2}(7 - \alpha)P$

$\uparrow Y = \sqrt{3}P\sin\frac{\pi}{6} + 2\sqrt{3}P + \alpha P\sin\frac{\pi}{3}$ (5)

$= \frac{\sqrt{3}}{2}P + 2\sqrt{3}P + \frac{\sqrt{3}}{2}\alpha P$

$= \frac{\sqrt{3}}{2}(5 + \alpha)P$

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$\tan\frac{\pi}{3} = \frac{Y}{X}$ (5)

$\therefore Y = \sqrt{3}X$

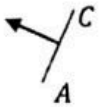
i.e. $\frac{\sqrt{3}}{2}(5+\alpha)P = \sqrt{3}.\frac{1}{2}(7-\alpha)P$

$\therefore \alpha = 1$ (5)

20

OR

(10)



(5)

$\alpha P\left(\frac{\sqrt{3}}{2}\right) + 2\sqrt{3}P\left(\frac{1}{2}\right) - \sqrt{3}P\left(\frac{1}{2}\right) - 2P\left(\frac{\sqrt{3}}{2}\right) = 0.$

$\Rightarrow \alpha = 1 + 2 - 2.$

$\Rightarrow \alpha = 1.$

(5)

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$R = \sqrt{3}P\left(\frac{\sqrt{3}}{2}\right) + 2P\left(\frac{1}{2}\right) + 2\sqrt{3}P\left(\frac{\sqrt{3}}{2}\right) + P\left(\frac{1}{2}\right)$ (10)

$= \frac{3P}{2} + \frac{2P}{2} + \frac{6P}{2} + \frac{P}{2}$

$= 6P.$

(5)

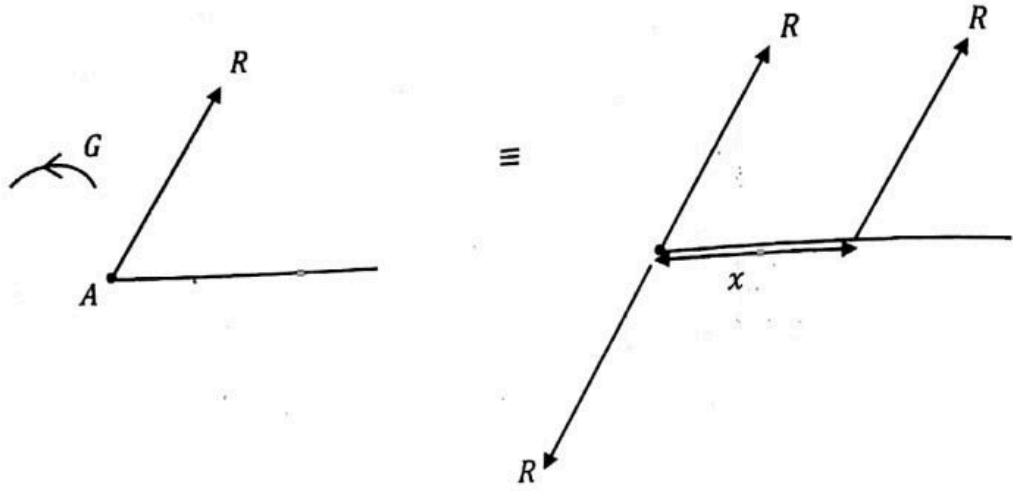
$\overleftarrow{A}$; $G = 2\sqrt{3}P.a + P\left(\frac{\sqrt{3}}{2}\right).2a$ (5)

$G = 2\sqrt{3}Pa\left(1 + \frac{1}{2}\right)$

$G = 3\sqrt{3}Pa$ (5)

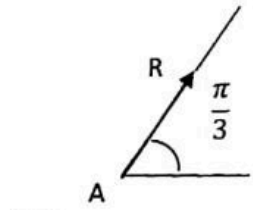
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Magnitude of the resultant = $R = 6P$ (5)

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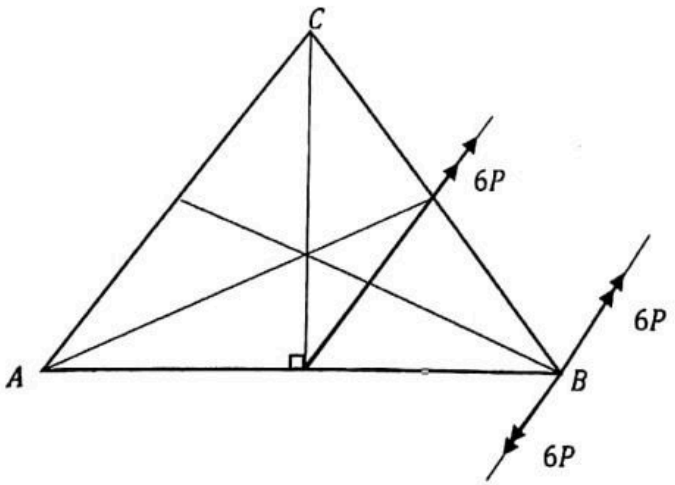


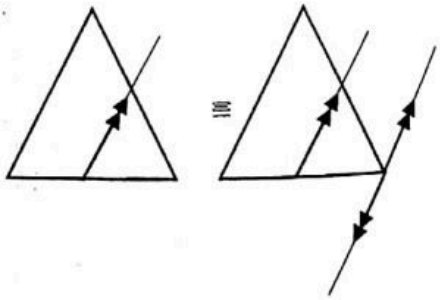
(5)

A ; $3\sqrt{3}Pa = 6P\left(\frac{\sqrt{3}}{2}\right)x$ (5)

$\therefore x = a$ (5)

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(5)

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$H = 6P.a\left(\frac{\sqrt{3}}{2}\right)$

$= 3\sqrt{3}Pa$ (5)

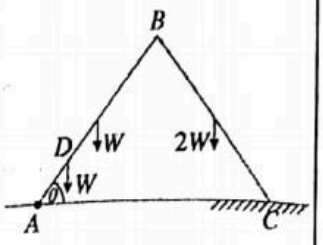
Contraclockwise sense (5)

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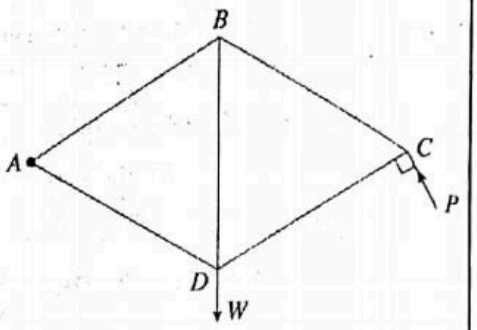
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15. (*a*) Two uniform rods AB and BC, each of length $2a$, are smoothly joined at the end B. The weights of the rods AB and BC are W and $2W$, respectively. The end A is smoothly hinged to a fixed point on a horizontal floor. A particle of weight W is attached to the point D on rod AB such that $AD = \frac{a}{2}$. The system is in equilibrium in a vertical plane such that $B\hat{A}C = \theta$ and the end-point C of the rod BC on a rough portion of the above horizontal floor, as shown in the figure. The coefficient of friction between the rod BC and the floor is μ. Show that $\cot\theta \leq \frac{15}{7}\mu$.

Find the reaction exerted on AB by CB at the joint B.



(*b*) The framework shown in the figure consists of five light rods AB, BC, CD, DA and DB of equal lengths smoothly jointed at their ends. A load W is suspended at the joint D and the framework is smoothly hinged at A to a fixed point and kept in equilibrium in a vertical plane with BD vertical by a force P applied to it at the joint C and perpendicular to the rod CD, in the direction shown in the figure.

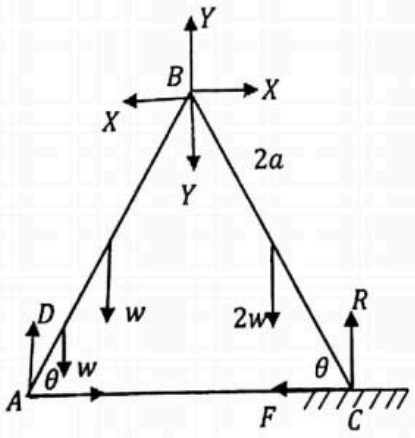


(i) Find the value of P.

(ii) Draw a stress diagram using Bow's notation for the joints C, B and D.

Hence, find the stresses in the rods, stating whether they are tensions or thrusts.

(a)



For the system;

$A\circlearrowright$

$$R.4a\cos\theta - w\left(\frac{a}{2}\cos\theta + a\cos\theta\right) - 2w(2a\cos\theta + a\cos\theta) = 0 \quad (15)$$

$$\therefore 4R = \frac{3}{2}w + 6w$$

$$R = \frac{15}{8}w.$$

For BC:

$B\circlearrowright$ $2wa\cos\theta + F2a\sin\theta - R.2a\cos\theta = 0$ (10)

$$\therefore w + F\tan\theta = R$$

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$\therefore F\tan\theta = \frac{15}{8}w - w.$

$\therefore F = \frac{7}{8}w\cot\theta.$ (5)

For the Equilibrium,

$$\mu \geq \frac{F}{R}.$$

$$\frac{7}{8}w\cot\theta \leq \mu\frac{15}{8}w$$

$$\cot\theta \leq \frac{15}{7}\mu.$$ (5)

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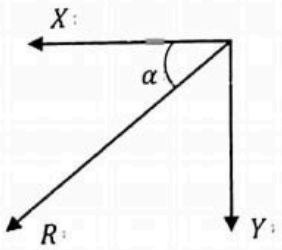
$\leftarrow BC$: $X = F = \frac{7}{8}w\cot\theta$ (5)

$\uparrow R + Y = 2w$ (5)

$Y = 2w - R$

$= 2w - \frac{15}{8}w$

$= \frac{w}{8}$ (5)



$R^2 = X^2 + Y^2$

$= \frac{49}{64}w^2\cot^2\theta + \frac{w^2}{64}$

$R = \frac{w}{8}\sqrt{1 + 49\cot^2\theta}$ (5)

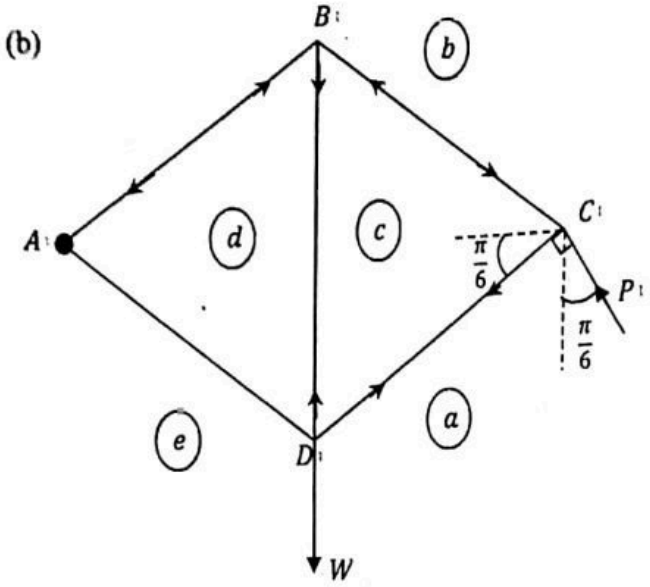
$\tan\alpha = \frac{Y}{X} = \frac{w/8}{7w/8\cot\theta} = \frac{\tan\theta}{7}$

$\alpha = \tan^{-1}\left(\frac{\tan\theta}{7}\right)$ (5)

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(b)

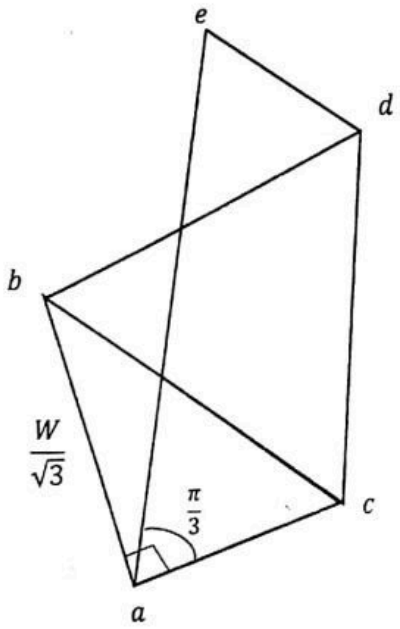


A $P\cos\frac{\pi}{6}.2x - Wx = 0$ (5)

(Here $AC = 2x$)

$\therefore P = \frac{W}{\sqrt{3}}$ (5)

10



(10) + (10) + (10)

Each Joint (10)

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Rod	Tension	Thrust
AB		$\frac{2W}{3}$
BC		$\frac{2W}{3}$
CD	$\frac{W}{3}$	
DA	$\frac{W}{3}$	
BD	$\frac{2W}{3}$	

magnitude - (5) each

Tension/Thrust (15)

All 5 correct (15)

4 correct (10)

3 correct (5)

50

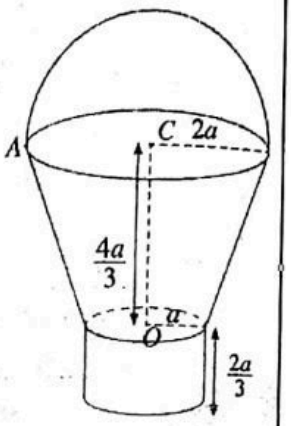
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16. Show that the centre of mass of

(i) a thin uniform wire in the shape of a semi-circular arc of radius a, is at a distance $\frac{2a}{\pi}$ from its centre,

(ii) a uniform hollow right circular cone of height h is at a distance $\frac{1}{3}h$ from the centre of the base of the cone.

A bucket is made by rigidly fixing to a uniform thin shell in the shape of a frustum of hollow right circular cone of radii of the upper and lower circular rims $2a$ and a, respectively and height $\frac{4a}{3}$ the following parts at the places each meets this shell as shown in the figure:



- A uniform thin circular plate of radius a and centre at O.
- A uniform thin shell in the shape of a hollow right circular cylinder of radius a and height $\frac{2a}{3}$.
- A uniform thin wire in the shape of a semi-circle of radius $2a$ and centre at C.

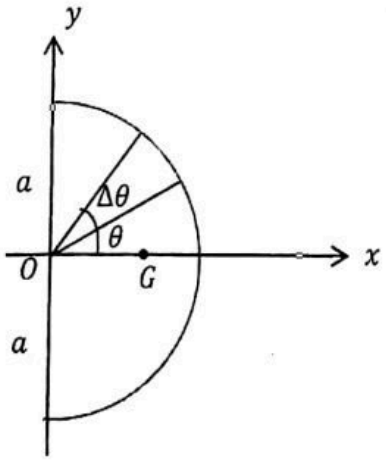
The mass per unit area of the frustum, plate and the cylinder is σ and the mass per unit length of the wire is $11a\sigma$.

Show that the distance from O to the centre of mass of the bucket is $(10\pi+27)\frac{a}{9\pi}$.

Find the angle OC makes with the downward vertical in the equilibrium position, when the bucket is hanged freely by a vertical string from the point A at which the wire meets the upper rim of the frustum.

(i)

Semi-circular wire



By Symmetry, centre of mass G lies on x –axis. (5)

$\Delta m = a\Delta\theta\rho$, where ρ is the mass per unit length.

Let $OG = \bar{x}$.

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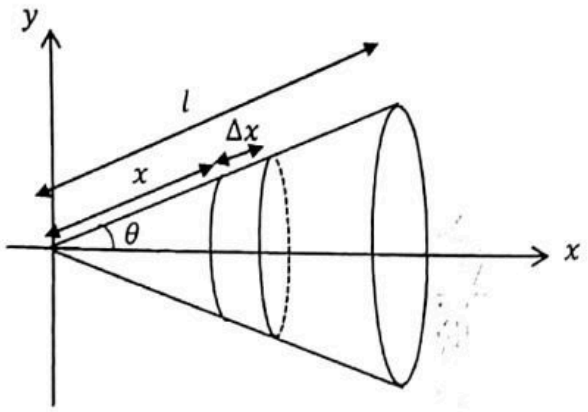
Then

$$\bar{x} = \frac{\int_{-\pi/2}^{\pi/2} a\rho\, a\cos\theta\, d\theta}{\int_{-\pi/2}^{\pi/2} a\rho\, d\theta} = \frac{a\sin\theta \Big|_{-\pi/2}^{\pi/2}}{\theta \Big|_{-\pi/2}^{\pi/2}} = \frac{2a}{\pi}$$

(5) (5) (5) (5) (5)

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(ii)



By Symmetry, centre of mass G lies on x −axis. (5)

$$h = l\cos\theta$$

$\Delta m = 2\pi(x\sin\theta)\Delta x\sigma$, where σ is the mass per unit length.

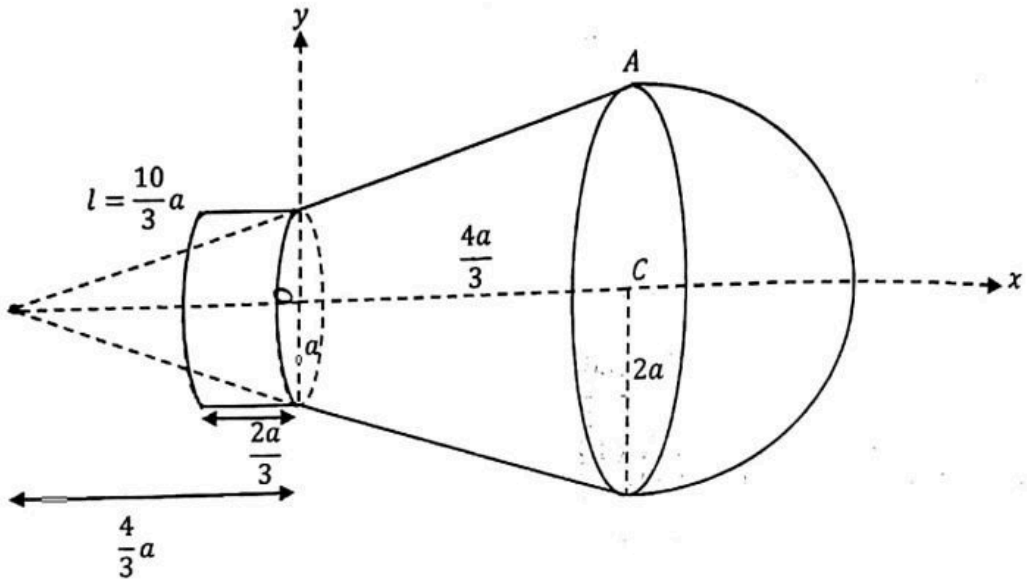
$$\therefore \bar{x} = \frac{\int_0^l x\cos\theta\, 2\pi\sigma x\sin\theta\, dx}{\int_0^l 2\pi\sigma x\sin\theta\, dx} = \frac{\cos\theta\int_o^l x^2 dx}{\int_o^l x\, dx} = \frac{h/2\,\frac{x^3}{3}\Big|_0^l}{\frac{x^2}{2}\Big|_0^l} = \frac{2h}{3}$$

(5) (5) (5) (5) (5)

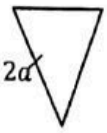
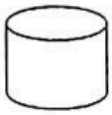
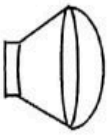
$\therefore$ The required distance $= \frac{h}{3}$.

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Object	Mass	Distance from $O(\uparrow)$
	$\pi(2a)(11a\sigma)$ $= 22\pi a^2\sigma$ (5)	$\frac{4}{3}a + 2\frac{(2a)}{\pi} = \frac{4}{3}a + \frac{4a}{\pi}$ (5)
	$\pi(2a)\left(\frac{10}{3}a\right)\sigma$ $= \frac{20}{3}\pi a^2\sigma$ (5)	$\left[\frac{2}{3}\left(\frac{8}{3}a\right) - \frac{4}{3}a\right] = \frac{4}{9}a$ (5)
	$\pi(a)\left(\frac{5}{3}a\right)\sigma$ $= \frac{5}{3}\pi a^2\sigma$ (5)	$-\frac{1}{3}\left(\frac{4}{3}a\right) = -\frac{4}{9}a$ (5)
	$2\pi a\left(\frac{2}{3}a\right)\sigma$ $= \frac{4}{3}\pi a^2\sigma$ (5)	$-\frac{1}{3}a$ (5)
	$\pi a^2\sigma$	0 (5)
	$22\pi a^2\sigma + \frac{20}{3}\pi a^2\sigma + \frac{5}{3}\pi a^2\sigma + \frac{4}{3}\pi a^2\sigma$ $= \frac{88}{3}\pi a^2\sigma$ (5)	$\bar{x}$

By symmetry, centre of mass lies on the x −axis. (5)

$$\frac{88}{3}\pi a^2\sigma\bar{x} = 22\pi a^2\sigma\left(\frac{4}{3}a + \frac{4a}{\pi}\right) + \frac{20}{3}\pi a^2\sigma\left(\frac{4}{9}a\right) - \frac{5}{3}\pi a^2\sigma\left(-\frac{4}{9}a\right) + \frac{4}{3}\pi a^2\sigma\left(-\frac{1}{3}a\right)$$

$$\frac{88}{3}\bar{x} = 4a\left(\frac{22}{3} + \frac{22}{\pi} + \underbrace{\frac{20}{27} + \frac{5}{27} - \frac{1}{9}}_{\frac{22}{27}}\right) \quad (15)$$

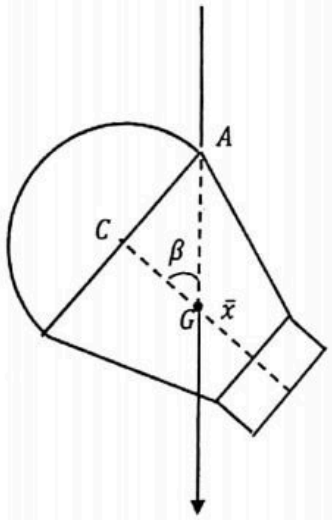
$$\frac{88}{3}\bar{x} = 22 \times 4a\left(\frac{10}{27} + \frac{22}{\pi}\right)$$

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$$\frac{88}{3}\bar{x} = 88a\left(\frac{(10\pi + 27)}{27\pi}\right)$$

$$\bar{x} = \frac{a}{9\pi}(10\pi + 27) \quad (5)$$

75



(5)

$$\tan\beta = \frac{AC}{CG} = \frac{2a}{\frac{4}{3}a - \bar{x}} \quad (5)$$

$$= \frac{18\pi}{27 - 2\pi} \quad (5)$$

$$\therefore \beta = \tan^{-1}\left(\frac{18\pi}{27 - 2\pi}\right)$$

15

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17.(a) Two identical boxes A and B, each contains 10 balls which are identical in all respects except for their colour. The box A contains 6 white balls and 4 red balls, and the box B contains 8 white balls and 2 red balls. A box is chosen at random and 3 balls are drawn from that box at random, one after the other, without replacement. Find the probability that

(i) two red balls and one white ball are drawn,

(ii) the box A was chosen, given that two red balls and one white ball are drawn.

(b) Let the mean and the standard deviation of the set of data $\{x_1, x_2, \ldots, x_n\}$ be $\bar{x}$ and σ_x respectively, and let $y_i = \frac{x_i - \alpha}{\beta}$ for $i = 1, 2, \ldots, n$ where α and β (>0) are real constants. Show that $\bar{y} = \frac{\bar{x} - \alpha}{\beta}$ and $\sigma_y = \frac{\sigma_x}{\beta}$, where $\bar{y}$ and σ_y are respectively the mean and the standard deviation of the set of data $\{y_1, y_2, \ldots, y_n\}$.

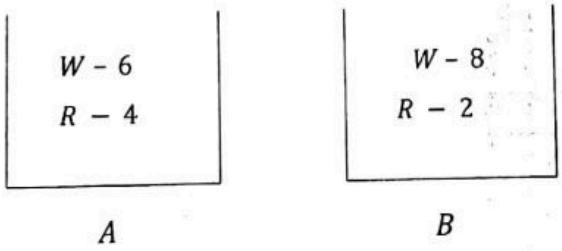
Monthly instalments for an insurance scheme by 100 employees of a company are given in the following frequency table:

Monthly Instalment (rupees) x	Number of employees
1500 – 3500	30
3500 – 5500	40
5500 – 7500	20
7500 – 9500	10

By means of the transformation $y = \frac{x - 500}{1000}$, estimate the mean and the standard deviation of y, and also the coefficient of skewness of y defined by $\frac{3(\text{mean} - \text{median})}{\text{standard deviation}}$.

Hence, estimate the mean, the standard deviation and the coefficient of skewness of x.

(a)



Let X be the event that two red balls and one white ball are drawn.

(i) $P(X) = P(X|A)P(A) + P(X|B)P(B)$ _____(1) (5)

$P(A) = P(B) = \frac{1}{2}$. (5)

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(5) (5) (5)

$$P(X|A) = \frac{4}{10}\times\frac{3}{9}\times\frac{6}{8} + \frac{4}{10}\times\frac{6}{9}\times\frac{3}{8} + \frac{6}{10}\times\frac{4}{9}\times\frac{3}{8}$$

$$= \frac{3}{10}. \quad (5)$$

(5) (5) (5)

$$P(X|B) = \frac{2}{10}\times\frac{1}{9}\times\frac{8}{8} + \frac{2}{10}\times\frac{8}{9}\times\frac{1}{8} + \frac{8}{10}\times\frac{2}{9}\times\frac{1}{8}$$

$$= \frac{1}{15}. \quad (5)$$

Now (1) given as,

$$P(X) = \frac{3}{10}\times\frac{1}{2} + \frac{1}{15}\times\frac{1}{2} = \frac{11}{60} \quad (5)$$

55

(ii)

$$P(A|X) = \frac{P(X|A)P(A)}{P(X)} \quad (5) \quad \text{[or Bayes' theorem]}$$

$$= \frac{\frac{3}{10}\times\frac{1}{2}}{\frac{11}{60}}$$

$$= \frac{9}{11} \quad (5)$$

10

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(b)

$$y_i = \frac{x_i - \alpha}{\beta}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \quad (5)$$

$$= \frac{1}{n\beta} \sum_{i=1}^{n} (x_i - \alpha)$$

$$= \frac{1}{n\beta} \left\{ \sum_{i=1}^{n} x_i - n\alpha \right\}$$

$$= \frac{1}{\beta} \left\{ \frac{\sum_{i=1}^{n} x_i}{n} - \alpha \right\}$$

$$= \frac{\bar{x} - \alpha}{\beta}. \quad (5)$$

$$\sigma_y^2 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n} \quad (5)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_i - \alpha}{\beta} - \frac{\bar{x} - \alpha}{\beta} \right)^2$$

$$= \frac{1}{n\beta^2} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \frac{\sigma_x^2}{\beta^2}$$

$$\therefore \sigma_y = \frac{\sigma_x}{\beta} \quad (5) \quad (\because \beta > 0)$$

20

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(5) (5) (5)

Class Interval x	f	Class Interval y	Mid-point y	fy	fy^2
1500-3500	30	1-3	2	60	120
3500-5500	40	3-5	4	160	640
5500-7500	20	5-7	6	120	720
7500-9500	10	7-9	8	80	640
				$\sum fy = 420$	$\sum fy^2 = 2120$

(5) (5)

$$\bar{y} = \frac{\sum fy}{\sum f} = \frac{420}{100} = 4.2 \quad (5)$$

(5)

$$\sigma_y = \sqrt{\frac{\sum fy^2}{\sum f} - \bar{y}^2} = \sqrt{\frac{2120}{100} - 4.2^2}$$

$$= \sqrt{21.2 - 17.64}$$

$$= \sqrt{3.56} \approx 1.887$$

(5)

Let M_y = Median of $y = 50^{\text{th}}$ data

Then

$$M_y = 3 + \frac{(50-30)}{40}(5-3) = 4 \quad (5)$$

∴ The coefficient of skewnes: $y \approx \frac{3(4.2-4)}{\sqrt{3.56}} \approx 0.317$

(5) 50

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$\bar{x} = 1000\bar{y} + 500$

$= 1000 \times 4.2 + 500$

$= 4700$ (5)

$\sigma_x = 1000\ \sigma_y$

$\approx 1000 \times 1.887$

$= 1887$ (5)

The coefficient of skewness does not change.

$S_x = S_y \approx 0.317$ (5)

15

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