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General Certificate of Education (Adv.Level) Examination, August 2017
Combined Mathematics

Paper I

Part A

1. for n = 1 L. H. S = 1.(3+1) = 4 and R. H. S = 1. $(1+1)^2 = 4$ ∴ The result is true for n = 1 (05)

Take any $P \in \mathbb{Z}^+$ and assume that the result is true for n = p

i.e $\sum_{r=1}^{p} r(3r+1) = P(P+1)^2$ ——————— (1) (05)

Now $\sum_{r=1}^{P+1} r(3r+1) = \sum_{r=1}^{P} r(3r+1) + (P+1)(3P+4)$ (05)

$= P(P+1)^2 + (P+1)(3P+4)$

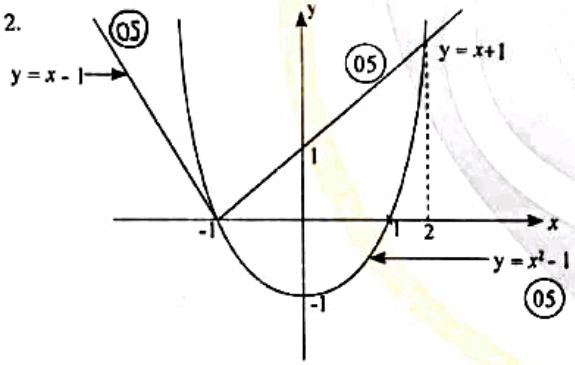
$= (P+1)(P^2+P+3P+4)$

$= (P+1)(P+2)^2$ (05)

Hence if the result is true for n = P, then it is also true for n = P+1, We have already proved that the result is true for n = 1

Hence by the Principle of mathematical Induction the result is true for all $n \in \mathbb{Z}^+$ (05) [25]

2. (05)



At the Points of intersection, We must have $x \geq -1$ (05)

and $x^2 - 1 = x + 1$, and so $x = -1$ or $x = 2$ The Solutions are the values of x Satisfying $x \leq -1$ or $x \geq 2$ (05) [25]

Aliter 1

$$|x+1| = \begin{cases} x+1 & \text{if } x \geq -1 \\ -(x+1) & \text{if } x < -1 \end{cases}$$

Case (i) $x \geq -1$

In this case, $x^2 - 1 \geq x+1$ (05)

$\Leftrightarrow x^2 - x - 2 \geq 0$

$\Leftrightarrow (x+1)(x-2) \geq 0$

$\Leftrightarrow x \leq -1$ or $x \geq 2$ (05)

Since $x \geq -1$, the solutions are $x = -1$ or $x \geq 2$

Case (ii) $x < -1$

In this case, $x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq -(x+1)$ (05)

$\Leftrightarrow x^2 + x \geq 0$

$\Leftrightarrow x(x+1) \geq 0$

$\Leftrightarrow x \leq -1$ or $x \geq 0$ (05)

Since $x < -1$, the Solutions are $x < -1$ From the two cases, we get $x \leq -1$ or $x \geq 2$ as the answer. (05) [25]

Aliter 2

Case (i) $x > -1$

$x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq x+1$ (05)

$\Leftrightarrow x \leq -1$ or $x \geq 2$ (05)

Since $x > -1$, the Solutions are $x \geq 2$

Case (ii) $x \leq -1$

$x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq -(x+1)$ (05)

$\Leftrightarrow x \leq -1$ or $x \geq 0$ (05)

Since $x \leq -1$, the Solutions are $x \leq -1$ From the two cases, we get $x \leq -1$ or $x \geq 2$ as the answer. (05) [25]

Aliter 3

Case (i) $x^2 \geq 1$

In this case $x^2 - 1 \geq 0$, and so both sides are non - negative

$\therefore x^2 - 1 \geq |x+1|$

$\Leftrightarrow (x^2 - 1)^2 \geq (x+1)^2$ (05)

$\Leftrightarrow (x+1)^2(x-1)^2 - (x+1)^2 \geq 0$

$\Leftrightarrow (x+1)^2[(x-1)^2 - 1] \geq 0$

$\Leftrightarrow (x+1)^2 x(x-2) \geq 0$ (05)

$\Leftrightarrow x = -1$ or $x \leq 0$ or $x \geq 2$ (05)

Since $x^2 \geq 1 \Leftrightarrow x < -1$ or $x \geq 1$ the Solutions are $x \leq -1$ or $x \geq 2$ (05)

Case (ii) $x^2 < 1$

Since $x^2 - 1 < 0$ and hence there are no Solution, From the two cases, we get $x \leq -1$ or $x \geq 2$ as the answer. (05) [25]

3. $(\sqrt{3}+5i) - 2i = \sqrt{3} + 3i$

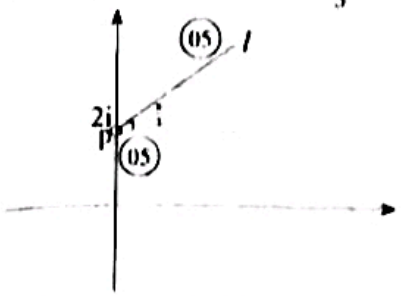
$= 2\sqrt{3}\left\{\frac{1}{2} + i\frac{\sqrt{3}}{2}\right\}$ (05)

$= 2\sqrt{3}\left\{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right\}$

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$\therefore PQ = 2\sqrt{3}$ (05)

Also, $\text{Arg}[(\sqrt{3}+5i)-2i] = \frac{\pi}{3}$ and hence Q lies on l. (05)



[25]

4. I N F T Y

3 2 1 1 1

$\frac{8!}{3!\,2!} = \frac{8\times7\times6\times5\times4}{2}$ (05)

$= 3360$ (05)

(I) $\frac{6!}{2!} = 360$ (05)

(II) I F T Y

2 1 1 1

$\frac{5!}{2!}\times 3 = 5\times4\times3\times3$

(05) $= 180$ (05)

[25]

5. $\lim_{x\to\alpha}\frac{x^3-\alpha^3}{\tan x-\tan\alpha} = \lim_{x\to\alpha}\frac{(x-\alpha)(x^2+\alpha x+\alpha^2)}{\frac{\sin x}{\cos x}-\frac{\sin\alpha}{\cos\alpha}}$ (05)

$= \lim_{x\to\alpha}\frac{(x-\alpha)\cos x\cos\alpha\,(x^2+x\alpha+\alpha^2)}{\sin x\cos\alpha-\cos x\sin\alpha}$ (05)

$= \lim_{x\to\alpha}\frac{x-\alpha}{\sin(x-\alpha)}\cdot\cos x\cos\alpha(x^2+\alpha x+\alpha^2)$ (05) (05)

$= 1.\cos\alpha\cos\alpha(3\alpha^2)$

$= 3\alpha^2\cos^2\alpha$ (05)

[25]

Aliter 1

$\lim_{x\to\alpha}\frac{x^3-\alpha^3}{\tan x-\tan\alpha} = \lim_{x\to\alpha}\frac{(x-\alpha)(x^2+x\alpha+\alpha^2)}{\tan(x-\alpha)(1+\tan x\tan\alpha)}$ (05)

$= \lim_{x\to\alpha}\frac{x-\alpha}{\tan(x-\alpha)}\cdot\frac{x^2+\alpha x+x^2}{(1+\tan x\tan\alpha)}$ (05)

$= \lim_{x\to\alpha}\frac{x-\alpha}{\sin(x-\alpha)}\cdot\frac{\cos(x-\alpha).(x^2+\alpha x+\alpha^2)}{1+\tan x\tan\alpha}$ (05)

$= \frac{1.1\;3\alpha^2}{1+\tan^2\alpha}$ (05)

$= \frac{3\alpha^2}{\sec^2\alpha} = 3\alpha^2\cos^2\alpha$ (05)

[25]

Aliter 2 (05)

$\lim_{x\to\alpha}\frac{x^3-\alpha^3}{\tan x-\tan\alpha} = \lim_{x\to\alpha}\frac{x^3-\alpha^3}{x-\alpha}\cdot\frac{x-\alpha}{\frac{\sin x}{\cos x}-\frac{\sin\alpha}{\cos\alpha}}$

$= \lim_{x\to\alpha}\frac{x^3-\alpha^3}{x-\alpha}\cdot\frac{x-\alpha}{\frac{\sin x\cos\alpha-\cos x\sin\alpha}{\cos x\cos\alpha}}$ (05)

$= \lim_{x\to\alpha}\frac{x^3-\alpha^3}{x-\alpha}\cdot\frac{(x-\alpha)}{\sin(x-\alpha)}\cos x\cos\alpha$ (05)

$= 3\alpha^2\cdot1\cdot\cos^2\alpha$ (05)

$= 3\alpha^2\cos^2\alpha$ (05)

[25]

6. $\frac{d}{dx}\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)$

$= \frac{1}{\sqrt{1-\frac{(b-a)}{b}\cos^2x}}\times\sqrt{\frac{b-a}{b}}\;(-\sin x)$ (05) (05)

$= \frac{-\sin x}{\sqrt{b-b\cos^2x+a\cos^2x}}\times\sqrt{b-a}$

$= \frac{-\sin x}{\sqrt{b-b\cos^2x+a\cos^2x}}\times\sqrt{b-a}$

$= \frac{-\sqrt{b-a}\,\sin x}{\sqrt{a\cos^2x+b\sin^2x}}$ (05)

$\int\frac{-\sqrt{b-a}\,\sin x}{\sqrt{a\cos^2x+b\sin^2x}}dx = \sin^{-1}\left[\sqrt{\frac{b-a}{b}}\cos x\right]+C$ (05)

$\int\frac{\sin x}{\sqrt{a\cos^2x+b\sin^2x}}dx = \frac{-1}{\sqrt{b-a}}\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)+C$

Where C is an arbitrary constant. (05) [25]

Aliter

Let $y = \sin^{-1}\left[\sqrt{\frac{b-a}{b}}\cos x\right]$

Then $\sin y = \sqrt{\frac{b-a}{b}}\cos x$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

$\cos y\frac{dy}{dx} = \sqrt{\frac{b-a}{b}}(-\sin x)$ ——— (1) (05)

$\cos y = \sqrt{1-\sin^2 y}\;\left(\because -\frac{\pi}{2}\le y\le\frac{\pi}{2}\right)$

$= \sqrt{1-\frac{b-a}{b}\cos^2x}$

$= \sqrt{\frac{b(1-\cos^2x)+a\cos^2x}{b}}$

$= \frac{\sqrt{a\cos^2x+b\sin^2x}}{\sqrt{b}}$ (05)

(1) $\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{b-a}\,\sin x}{\sqrt{a\cos^2x+b\sin^2x}}$ (05)

Integration as before (10) [25]

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7. $x = 3\cos\theta - \cos^3\theta \quad y = 3\sin\theta - \sin^3\theta$ (05)

$\frac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta\sin\theta$ (05)

$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \frac{3\cos\theta(1-\sin^2\theta)}{-3\sin\theta(1-\cos^2\theta)} = \frac{\cos^3\theta}{\sin^3\theta} = -\cot^3\theta$ (05)

$\frac{dy}{dx} = -1 \Leftrightarrow \cot\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ (05)

$p = \left(\frac{3}{\sqrt{2}} - \frac{1}{2\sqrt{2}}, \frac{3}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\right) = \left(\frac{5}{2\sqrt{2}}, \frac{5}{2\sqrt{2}}\right)$ (05)

25

8. Bisectors are given by

$\frac{3x-4y-2}{5} = \pm\frac{4x-3y-1}{5}$ (05)

$x + y + 1 = 0$ or $7x - 7y - 3 = 0$ (05)

Let α be the acute angle between t_1 and $x + y + 1 = 0$

$\tan\alpha = \left|\frac{\frac{3}{4}+1}{1-\frac{3}{4}}\right| = 7 > 1$ (05) (05)

$\therefore 7x - 7y - 3 = 0$ is the bisector of the acute angle between t_1 and t_2 (05) 25

9. The required equation has the from $(x^2 + y^2 - 4) + \lambda(y - x - 1) = 0$, where $\lambda \in R$

i.e $x^2 + y^2 - \lambda x + \lambda y - \lambda - 4 = 0$ (10)

If this is orthogonal to s, with $g = 0, f = 0, c = -4, g' = -\frac{\lambda}{2}$ $f' = \frac{\lambda}{2}, c' = -\lambda - 4$ we must have $2gg' + 2ff' = c + c'$ (05)

i. e. $0 = -\lambda - 8$

$\therefore \lambda = -8$ (05)

$\therefore$ The answer is $x^2 + y^2 + 8x - 8y + 4 = 0$ (05) 25

10. $\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2 = \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2}$ (05)

$= 1 + \sin\theta \quad \left(\because \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1 \text{ and } 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta\right)$

Let $\theta = +\frac{\pi}{6}$ (05)

Then $\left(\cos\frac{\pi}{12} + \sin\frac{\pi}{12}\right)^2 = 1 + \frac{1}{2}$

$\therefore \sin\frac{\pi}{12} + \cos\frac{\pi}{12} = \sqrt{\frac{3}{2}}$ ——① $(\because \sin\frac{\pi}{12} + \cos\frac{\pi}{12} > 0)$ (05)

Let $\theta = -\frac{\pi}{6}$

Then $\left(\cos\frac{\pi}{12} - \sin\frac{\pi}{12}\right)^2 = \frac{1}{2}$ (05)

$\therefore \cos\frac{\pi}{12} - \sin\frac{\pi}{12} = \frac{1}{\sqrt{2}}$ ——② $(\because \sin\frac{\pi}{12} < \cos\frac{\pi}{12})$

① - ② $\Rightarrow \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ (05) 25

Part B

11. (a) The discriminant $\Delta = (2a)^2 - 4.3.b$ (05)

$= 4(a^2 - 3b)$ (05)

Since $f(x) = 0$ has two real distant roots, we must have $\Delta > 0$ (05)

$\therefore a^2 > 3b$ (05) 20

$\alpha - \beta = -\frac{2a}{3}$ and $\alpha\beta = \frac{b}{3}$ (05) (05) 10

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ (10)

$= \frac{4a^2}{9} - \frac{4b}{3}$ (05)

$= \frac{4}{9}(a^2 - 3b)$ (05)

$|\alpha - \beta| = \frac{2}{3}\sqrt{a^2 - 3b}$ (05) 2

Let $\alpha' = |\alpha+\beta|$ and $\beta' = |\alpha-\beta|$

Then $\alpha' = \frac{2}{3}|a|$ and $\beta' = \frac{2}{3}\sqrt{a^2 - 3b}$ (05)

The equired equation is $(x - \alpha')(x - \beta') = 0$ (05)

i.e. $x^2 - (\alpha' + \beta')x + \alpha'\beta' = 0$ (05)

$\Rightarrow x^2 - \left(\frac{2}{3}|a| + \frac{2}{3}\sqrt{a^2 - 3b}\right)x + \frac{4}{9}|a|\sqrt{a^2 - 3b} = 0$ (05)

$\Rightarrow 9x^2 - 6\left(|a| + \sqrt{a^2 - 3b}\right)x + 4\sqrt{a^4 - 3a^2b} = 0$ (05) 30

(b) Since the remainder when g (x) divided by $(x - 1)(x + 2)$ is $3x + 2$, We have.

$g(x) = h(x)(x - 1) + 3x + 2$ ——① (10)

Where $h(x)$ is a polynominal of degree 1 by the Remainder Theorem, the remaimder when g (x) is divided by $(x - 1)$ is g (1) (05)

(1) $\Rightarrow$ g (1) = 5 (05)

Hence the remainder when g (x) divided by $(x - 1)$ is Again by the remainder theorem the remainder when g (x) is divided by $(x + 2)$ is g (-2) (05)

① $\Rightarrow$ g (-2) = -4 (05)

Hence the remainder when g (x) divided by $(x + 2)$ is -4.

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$g(1) = 5 \Rightarrow 1 + p + q + 1 = 5$ (05)

$p + q = 3$

$g(-2) = -4 \Rightarrow -8 + 4p - 2q + 1 = -4$ (05)

$4p - 2q = 3$

$p = \frac{3}{2}$ and $q = \frac{3}{2}$ (05) (05)

20

Now $g(-1) = -1 + p - q + 1 = 0$ $(\because p = q)$ (05) (05)

Thus by the factor theorem, $(x+1)$ is a factor of $g(x)$ (05)

15

12. (a) $(5+2x)^{14} = \sum_{r=0}^{14} {}^{14}C_r\, 5^{14-r} (2x)^r$ (10)

$= \sum_{r=0}^{14} {}^{14}c_r 5^{14-r} . 2^r . x^r$, where

${}^{14}c_r = \frac{14!}{r!(14-r)!}$ (05)

for $r = 0, 1, \ldots 14$

15

Let $T_r = {}^{14}C_r 5^{14-r} . 2^r . x^r$ for $r = 0, 1, \ldots\ldots 14$

Then $\frac{T_{(r+1)}}{T_r} = \frac{14!\, 5^{13-r} 2^{r+1}}{(r+1)!(13-r)!} x^{r+1} \Big/ \frac{14!\, 5^{14-r} 2^r x^r}{r!(14-r)!}$ (10) (05)

$= \frac{2(14-r)}{5(r+1)} x$ (05)

20

Thus $x = \frac{4}{3} \Rightarrow \frac{T_{r+1}}{T_r} = \frac{2}{5} \frac{(14-r)}{(r+1)} \frac{4}{3}$ (05)

Hence $\frac{T_{(r+1)}}{T_r} \gtrless 1$ according as $\frac{8}{15} \frac{(14-r)}{r+1} \gtrless 1$ (05)

i.e according as $112 - 8r \gtrless 15r + 15$ (05)

i.e according as $r \lessgtr \frac{97}{23} = 4\frac{5}{23}$ (05)

$T_0 < T_1 < T_2 < T_3 < T_4 < T_5 > T_6 \ldots\ldots\ldots > T_{14}$ (10)

$\therefore$ The required value is $r = 5$ (05)

35

(b) $\frac{1}{r+c} - \frac{1}{r+c+2} = \frac{(r+c+2) - (r+c)}{(r+c)(r+c+2)}$ (05)

$= \frac{2}{(r+c) \times (r+c+2)}$ (05)

10

Let $u_r = \frac{2}{(r+c) \times (r+c+2)}$ for $r \in \mathbb{Z}^+$

Then $r = 1$; $u_1 = \frac{1}{1+c} - \frac{1}{3+c}$ (05)

$r = 2$ $u_2 = \frac{1}{2+c} - \frac{1}{4+c}$

$r = 3$ $u_3 = \frac{1}{3+c} - \frac{1}{5+c}$ (05)

$r = n-2$ $u_{n-2} = \frac{1}{n-2+c} + \frac{1}{n+c}$ (05)

$r = n-1$; $u_{n-1} = \frac{1}{n-1+c} - \frac{1}{n+c+1}$ (05)

$r = n$ $u_n = \frac{1}{n+c} - \frac{1}{n+c+2}$

$\sum_{r=1}^{n} u_r = \frac{1}{1+c} + \frac{1}{2+c} - \frac{1}{n+c+1} - \frac{1}{n+c+2}$ (10)

$= \frac{3+2c}{(1+c)(2+c)} - \frac{1}{n+c+1} - \frac{1}{n+c+2}$ (05) 35

The limit of the R. H.S as $n \to \infty$ is $\frac{3+2c}{(1+c)(2+c)}$ (10)

$\sum_{r=1}^{\infty} u_r$ convergent and the sum is $\frac{3+2c}{(1+c)(2+c)}$ (05) (05) 20

Put $c = 0$; $\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{3}{4}$ ——— (1) (05)

Put $c = 1$; $\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)} = \frac{5}{12}$ (05)

$\Rightarrow \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)} = \frac{1}{3} + \frac{5}{12} = \frac{3}{4}$ ——— (2)

Now (1) and (2) $\Rightarrow \sum_{r=1}^{\infty} \frac{1}{r(r+2)}$

$= \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$ (05)

15

13. (a) $AB^T = \begin{pmatrix} 2 & a & 3 \\ -1 & b & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & b \\ a & 0 \end{pmatrix}$ (05)

$= \begin{pmatrix} 2 - a + 3a & 2 + ab \\ -1 - b + 2a & -1 + b^2 \end{pmatrix}$ (10)

$AB^T = P \Leftrightarrow \begin{pmatrix} 2 - a + 3a & 2 + ab \\ -1 - b + 2a & -1 + b^2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$ (05)

$\Leftrightarrow 2 + 2a = 4, 2 + ab = 1, -1 + 2a - b = 2, -1 + b^2 = 0$ (10)

$\Leftrightarrow a = 1, \; b = -1$ (05)

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Now, $B^TA = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ -1 & -1 & 2 \end{pmatrix}$ (05)

$= \begin{pmatrix} 1 & 0 & 5 \\ -1 & 0 & -5 \\ 2 & 1 & 3 \end{pmatrix}$ (05) [45]

$P^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -2 & 4 \end{pmatrix}$ (10)

Also $PQ = P^2 + 2I \iff P^{-1}(PQ) = P^{-1}(P^2 + 2I)$ (05)

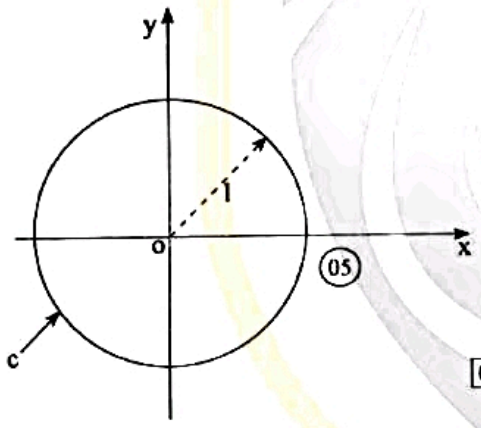
$\iff Q = P^{-1}P^2 + P^{-1}(2I)$ (05)

$\iff Q = P + 2P^{-1}$ (05)

$\iff Q = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & -4 \end{pmatrix}$ (05)

$Q = \begin{pmatrix} 4 & 2 \\ 4 & -4 \end{pmatrix}$ (05) [35]

(b)



(05) [05]

First $\frac{1}{z_0} = \frac{1}{a(\cos\theta + i\sin\theta)} \cdot \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}$ (05)

$= \frac{\cos\theta - i\sin\theta}{a(\cos^2\theta + \sin^2\theta)}$

$= \frac{1}{a}\cos(-\theta) + i\sin(-\theta)$ (05)

Hence $\left|\frac{1}{z_0}\right| = \frac{1}{a}$, and $\text{Arg}\left(\frac{1}{z_0}\right) = -\theta$ (05) (05)

Next $z_0^2 = a^2(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$

$= a^2[(\cos^2(\theta) - \sin^2\theta) + 2i\cos\theta\sin\theta]$ (05)

$= a^2(\cos 2\theta + i\sin 2\theta)$ (05)

Hence $= |z_0^2| = a^2$, and $\text{Arg}(z_0^2) = 2\theta$ (05) (05) [40]

Suppose that P lies on C

Then $a = 1$ (05)

$\therefore |\frac{1}{z_0}| = 1$ and so Q lies on C (05)

Also $|z_0^2| = 1$ and hence S lies on C (05) [15]

$z_0 + \frac{1}{z_0} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$

$= 2\cos\theta$ (05)

Note that $0 < \theta < \frac{\pi}{2} \Rightarrow 0 < 2\cos\theta < 2$

$\therefore$ The number represented by $z_0 + \frac{1}{z_0}$ is real and lies between 0 and 2 on the real axis. (05) [10]

---

14. (a) $f(x) = \frac{x^2}{(x-1)(x-2)}$ for $x \neq 1, 2$

Then $f'(x) = \frac{(x-1)(x-2)2x - x^2(2x-3)}{(x-1)^2(x-2)^2}$ (10)

$= \frac{-6x^2 + 4x + 3x^2}{(x-1)^2(x-2)^2}$ (05)

$= \frac{x(4-3x)}{(x-1)^2(x-2)^2}$ for $x \neq 1, 2$ (05) [20]

Horizontal Asymptote: $\lim_{x \to \pm\infty} f(x) = 1$

Hence it is $y = 1$

Note that $\lim_{x \to 1^-} f(x) = \infty$ and $\lim_{x \to 1^+} f(x) = -\infty$

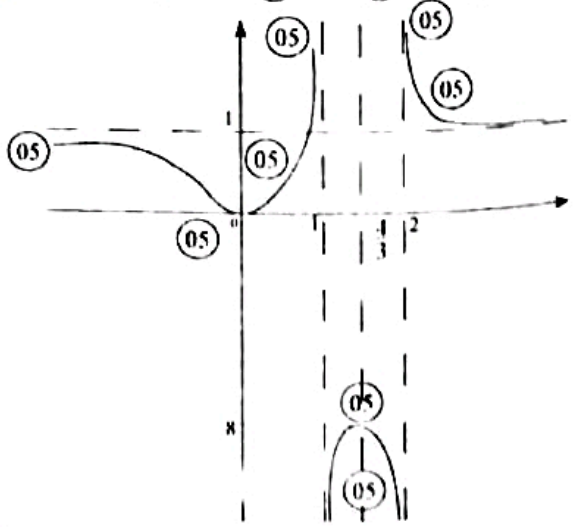
$\lim_{x \to 2^-} f(x) = -\infty$ and $\lim_{x \to 2^+} f(x) = \infty$

Vertical Asymptotes : $x = 1, 2$

$f'(x) = 0 \Leftrightarrow x = 0$ or $x = \frac{4}{3}$ (05)

	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \frac{4}{3}$	$\frac{4}{3} < x < 2$	$2 < x < \infty$
sign of $f'(x)$	( )	(+)	(+)	(−)	(−)

(05) (05) (05) (05) (05)



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There are two turning Points : (0, 0) Local minimum and $\left(\frac{4}{3}, -8\right)$ is a local maximum. 70

$x = 0$ or $2 < x < 2$
(05) (05) 10

(b) Area : $(5x)(3y) - 4xy = 385$ (05)

$11xy = 385$

$xy = 35$

$y = \frac{35}{x}$ (05)

Perimeter : $P = 2(5x+3y) + 4x + 4y$ (05)

$= 14x + 10y$

$= 14x + \frac{350}{x}$; $x > 0$ (05)

$\frac{dp}{dx} = 14 - \frac{350}{x^2}$ (05)

(05) $\frac{dp}{dx} = 0 \Leftrightarrow x^2 = \frac{350}{14} = 25$

$\therefore x = 5$ (05)

$\frac{dp}{dx} < 0$ for $0 < x < 5$ and $\frac{dp}{dx} > 0$ for $5 < x$
(05) (05)

$\therefore$ P is minimum when $x = 5$ (05)

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15 (i). $\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ (10)

$1 = A(x+1)^2 + Bx(x+1) + Cx$

$1 = (A+B)x^2 + (2A+B+C)x + A$

By comparing coefficients,

$x^0 : 1 = A$

$x^1 : 0 = 2A + B + C$ (10)

$x^2 : 0 = A + B$

$\therefore A = 1, B = -1$ and $C = -1$ (10)

$\int \frac{1}{x(x+1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx$ (05)

$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + c^1$ is an arbitrary constant. (15)

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(ii) $\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx$ (10)

$= -xe^{-x} - e^{-x} + c^{11}$, Where c^{11} is an arbitrary constant.
(05) (05)

Required area $= \int_1^2 xe^{-x}\,dx$ (05)

$= -(x+1)e^{-x}\Big]_1^2$ (05)

$= 2e^{-1} - 3e^{-2}$ (05) 35

(b) Let $x = c\tan\theta$

Then $dx = c\sec^2\theta\,d\theta$

when $x = 0$ $\theta = 0$ and when $x = c$, $Q = \frac{\pi}{4}$
(05)

Thus, I $= \int_0^{\frac{\pi}{4}} \frac{\ln c(1+\tan\theta)}{c^2 + c^2\tan^2\theta} \cdot c\sec^2\theta\,d\theta$
(05) (05)

$= \int_0^{\frac{\pi}{4}} \frac{\ln c(1+\tan\theta)}{c^2 \sec c^2\theta} \cdot c\sec^2\theta\,d\theta$ (05)

$= \frac{1}{c}\int_0^{\frac{\pi}{4}} (\ln c + \ln(1+\tan\theta))\,d\theta$ (05)

$= \frac{1}{c}\ln c\int_0^{\frac{\pi}{4}} d\theta + \frac{1}{c}\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$

$= \frac{1}{c}\ln c.[\theta]_0^{\frac{\pi}{4}} + \frac{1}{c}J$ (05)

$= \frac{\pi}{4c}\ln c + \frac{1}{c}J$ (05) 35

$J = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4c} - \theta\right)\right)d\theta$ (05)

$= \int_0^{\frac{\pi}{4}} \ln\left[1 + \frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$ (05)

$= \int_0^{\frac{\pi}{4}} \ln\frac{2}{(1+\tan\theta)}\,d\theta$

$= \int_0^{\frac{\pi}{4}} [\ln 2 - \ln(1+\tan\theta)]\,d\theta$ (05)

$= \ln 2.\frac{\pi}{4} - J$

$\therefore J = \frac{\pi}{8}\ln 2$ (05)

$$I = \frac{\pi}{4c} \ln C + \frac{1}{c}\frac{\pi}{8}\ln 2 \quad (05)$$

$$= \frac{\pi}{8c}(2\ln c + \ln 2)$$

$$= \frac{\pi}{8c}\ln(2C^2) \quad (05)$$

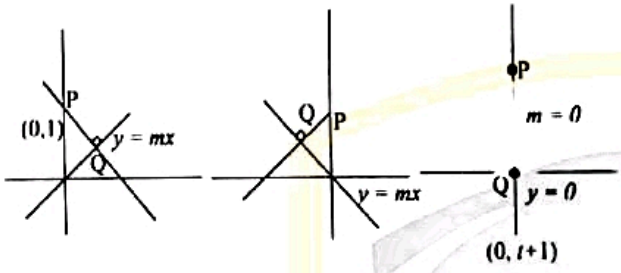
[30]

16. If the Point (0,1) lies on l, then we must have (05)

$1 = mx0$, i. e $1 = 0$, a contradiction

$\therefore$ (0,1) does not lie on l. (05)

[10]



Case (i) : $m \neq 0$

n this case the equation of the line through P erpendicular to l is given by

$$y - 1 = -\frac{1}{m}(x - 0) \quad (10)$$

Let us introduce t into this equation by

$y - 1 = -\frac{1}{m}(x - 0) = t$ (say) (05)

Then $y = t + 1$ and $x = -mt$, Where t is a parameter (05) (05)

Hence, the coordinates of any point on the line through P perpendicular to l can be written in the or $(-mt, t + 1)$, where t is a Parameter.

Case (ii) : $m = 0$

In this case the equation of the line through P perpendicular to l is the $y = ax$ is and hence the coordinates of any point on it can be written in the form $(0, t + 1)$ where t is a parameter. (05)

Thus, the form is true for all real values of m.

[30]

Let t_0 be the value of t corresponding to Q Since Q lies on l, $t_0 + 1 = m(-mt_0)$ (05)

$\therefore t_0 = -\frac{1}{1+m^2}$ (05), and hence $Q = \left[-m\left(-\frac{1}{1+m^2}\right), -\frac{1}{1+m^2}+1\right]$ (05)

$= \left[\frac{m}{1+m^2}, \frac{m^2}{1+m^2}\right]$ (05)

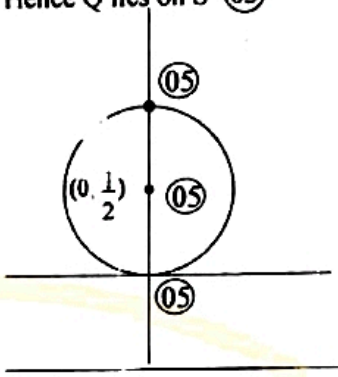
[20]

Put $x = \frac{m}{1+m^2}$ and $y = \frac{m^2}{1+m^2}$ in $x^2 + y^2 - y$: (05)

$$x^2 + y^2 - y = \frac{m^2}{(1+m^2)^2} + \frac{m^4}{(1+m^2)^2} - \frac{m^2}{1+m^2} \quad (05)$$

$$= \frac{m^2(1+m^2)}{(1+m^2)^2} - \frac{m^2}{1+m^2} = 0 \quad (05)$$

Hence Q lies on S (05)



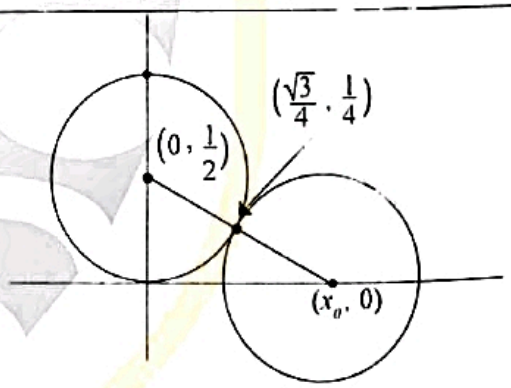
[35]

Put $x = \frac{\sqrt{3}}{4}$ and $y = \frac{1}{4}$ $\ln x^2 + y^2 - y$: (05)

$x^2 + y^2 - y = \frac{3}{16} + \frac{1}{16} - \frac{1}{4} = 0$ (05)

Heance $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right)$ lies ons (05)

[15]



Let x_0 be the x - coordinate of the centre of S', then $\sqrt{x_0^2 + \frac{1}{4}} = \frac{1}{2} + \sqrt{\left(\frac{\sqrt{3}}{4} - x_0\right)^2 + \frac{1}{16}}$ (05) (05) (05)

$$\Rightarrow x_0^2 + \frac{1}{4} = \frac{1}{4} + \sqrt{\left(\frac{\sqrt{3}}{4} - x_0\right)^2 + \frac{1}{16}} + \left(\frac{\sqrt{3}}{4} - x_0\right)^2 + \frac{1}{16} \quad (05)$$

$$\Rightarrow x_0 = \frac{\sqrt{3}}{2} \quad (05)$$

Hence the equation of s is $\left(x - \frac{\sqrt{3}}{2}\right)^2 + y^2 = \left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{1}{4}\right)^2$ (05)

i.e $\left(x - \frac{\sqrt{3}}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$

[30]

The equation of the requried circle touching S internally is

$$\left(x - \frac{\sqrt{3}}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2 \quad (10)$$

[10]

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17. (a) i $2\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right] - \frac{\cos\theta}{\sin\theta} = \sqrt{3}$ (05) (05) (05)

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ii Using the sine Rule : $\frac{AC}{\sin 80} = \frac{AB}{\sin 40}$ (05) (10)

$\Rightarrow \frac{AC}{AB} = \frac{2\sin 40^0 \cos 40^0}{\sin 40} = 2\cos 40$ (05) (05)

Again Using the sine Rule : $\frac{AC}{\sin(\alpha+20^0)} = \frac{AD}{\sin\alpha}$ (10)

$\Rightarrow \frac{AC}{AD} = \frac{\sin(20^0+\alpha)}{\sin\alpha}$ (05)

Hence, $AB = AD \Rightarrow \frac{\sin(20+\alpha)}{\sin\alpha} = 2\cos 40^0$ (05) (05)

$\therefore \sin(20^0+\alpha) = 2\sin\alpha\cos 40$

$\Rightarrow \sin 20^0\cos\alpha + \cos 20^0\sin\alpha = 2\sin\alpha\cos 40^0$ (05)

$\Rightarrow \cot\alpha = \frac{2\cos 40^0 - \cos 20^0}{\sin 20^0}$ (05)

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(i) with $\theta = 20^0 \Rightarrow \frac{2\cos 40^0 - \cos 20^0}{\sin 20^0} = \sqrt{3}$ (10) (05)

$\therefore \cot\alpha = \sqrt{3}$ (05)

$\Rightarrow \alpha = 30^0$ (Since $0^0 < \alpha < 90^0$) (05)

25

(b) $\cos 4x + \sin 4x = \cos 2x + \sin 2x$

$\Leftrightarrow \sin 4x - \sin 2x = \cos 2x - \cos 4x$ (05)

$\Leftrightarrow 2\cos 3x \sin x = 2\sin 3x \sin x$ (05) (05)

$\Leftrightarrow 2\sin x(\cos 3x - \sin 3x) = 0$ (05)

$\Leftrightarrow \sin x = 0$ or $\cos 3x = \sin 3x$ (05)

(05) (05) (05)

$\sin x = 0$ or $\tan 3x = 1$ $(\because \cos 3x \neq 0)$

$\Leftrightarrow x = n\pi$ for $n \in \mathbb{Z}$ or $3x = m\pi + \frac{\pi}{4}$ for $m \in \mathbb{Z}$ (05)

$\Leftrightarrow x = n\pi$ for $n \in \mathbb{Z}$ or $x = \frac{m\pi}{3} + \frac{\pi}{12}$ for $m \in \mathbb{Z}$ (05)

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