G.C.E. (Advanced Level) Examination - August 2016
Combined Mathematics - I
Scheme of Marking

Part A

(01) When n = 1, L.H.S $= \sum_{r=1}^{1} r(r+1) = 2$ and

$$\text{R.H.S} = {}^1/_3 (1+1)(1+2) = 2 \quad (05)$$

Hence, the result is true for n = 1

Take any $P \in \mathbb{Z}^+$ and assume that the result is true for n = P.

i. e $\sum_{r=1}^{P} r(r+1) = {}^P/_3 (P+1)(P+2)$ (05)

Now $\sum_{r=1}^{P+1} r(r+1) = \sum_{r=1}^{P} r(r+1) + (P+1)[(P+1)+1]$ (05)

$= {}^P/_3 (P+1)(P+2) + (P+1)(P+2)$

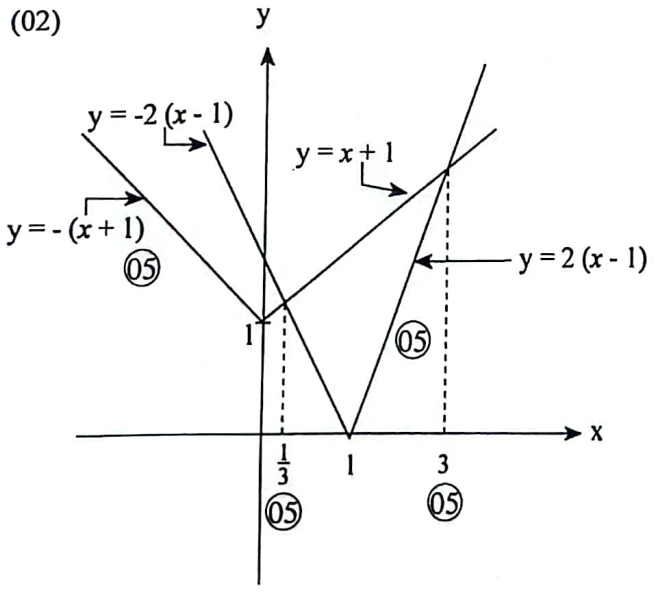
(By the induction hypothesis)

$= \frac{(P+1)}{3}(P+2)(P+3)$ (05)

Hence if the result is true for n = p, then it is also true for n = p + 1, We have already proved that the result is true for n = 1, Thus by the Principle of Mathematical Induction the result is true for all $n \in \mathbb{Z}^+$. (05)

[25]

(02)



From the diagram, $|x| + 1 > 2|x - 1| \Leftrightarrow {}^1/_3 < x < 3$ (05)

[25]

Aliter I

I. case (i) $\boxed{x \geqslant 1}$ $x + 1 > 2(x - 1) \Leftrightarrow x < 3$ (05)

Thus, in this case, x satisfying $1 \leqslant x < 3$ are the solutions.

II. case (ii) $\boxed{0 < x < 1}$ $x + 1 > -2(x - 1) \Leftrightarrow x > {}^1/_3$

Thus, in this case, x satisfying ${}^1/_3 < x < 1$ are the solutions. (05)

III. case (iii) $\boxed{x \leqslant 0}$ $-x + 1 > -2(x - 1) \Leftrightarrow x > 1$

This contradiction implies that there are no solutions in this case.

Thus the solution set is $\{x \in \mathbb{R} : {}^1/_3 < x < 3\}$ (05)

[15]

Aliter II

$|x| + 1 > 2|x - 1|$

$\Leftrightarrow x^2 + 2|x| + 1 > 4(x^2 - 2x + 1)$

$\Leftrightarrow 3x^2 - 2(|x| + 4x) + 3 < 0$

$\boxed{x > 0}$

$3x^2 - 10x + 3 < 0$ (05)

$\Leftrightarrow 1/3 < x < 3$ (05)

$\boxed{x < 0}$

$3x^2 - 6x + 3 < 0$

$\Leftrightarrow (x - 1)^2 < 0$ (05)

This is imposible. [15]

(03) Note that $O\hat{A}C = {}^{2\pi}/_3$

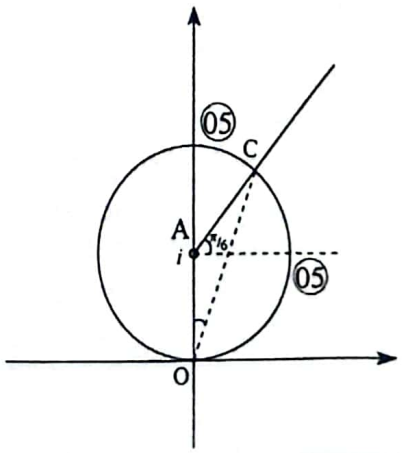
Since OA = AC, we have $A\hat{O}C = A\hat{C}O$

Hence $A\hat{O}C = {}^{\pi}/_6$ and so the angle OC makes with positive x - axis is ${}^{\pi}/_2 - {}^{\pi}/_6 = {}^{\pi}/_3$ (05)

Also $OC = 2\cos {}^{\pi}/_6 = \sqrt{3}$ (05)

Hence the required complex number;

$\sqrt{3}(\cos {}^{\pi}/_3 + i \sin {}^{\pi}/_3)$ (05)



[25]

Aliter

$y_c = 1 + \cos \pi/3 = 3/2$ (05)

$x_c = \sin \pi/3 = \frac{\sqrt{3}}{2}$ (05)

$\therefore 2_c = \frac{\sqrt{3}}{2} + i\,\frac{3}{2}$

$= \sqrt{3}\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$

$= \sqrt{3}(\cos \pi/3 + i \sin \pi/3)$ (05)

So $r = \sqrt{3}$ and $\theta = \pi/3$ [15]

(04) 5 4 3 2 1

The required answer = 5 ! (05)

= 120 (05)

(I) The last digit must either 2 or 4

The required answer = $2 \times 4 ! = 48$ (05)

(05)

(II) ③ ④ together in that order.

The number of ways = 4 !

④ ③ together in that order.

The number of ways is again 4 !

The required answer = $2 \times 4 !$ (05)

= 48 [25]

(05) $\lim_{x \to 0} \frac{1 - \cos(\alpha x)}{\sqrt{4 + x^2} - \sqrt{4 - x^2}}$ (05)

$= \lim_{x \to 0} \frac{\sin^2 \alpha x}{[1 + \cos(\alpha x)]} \frac{1}{(\sqrt{4 + x^2} - \sqrt{4 - x^2})} \frac{(\sqrt{4+x^2} + \sqrt{4-x^2})}{(\sqrt{4+x^2} + \sqrt{4-x^2})}$ (05)

$= \lim_{x \to 0} \frac{\sin^2 \alpha x}{2x^2} \frac{(\sqrt{4 + x^2} - \sqrt{4 - x^2})}{(1 + \cos(\alpha x)}$

$= \lim_{x \to 0} \left(\frac{\sin \alpha x}{\alpha x}\right)^2 \cdot \frac{\alpha^2}{2} \times \frac{(\sqrt{4 + x^2} + \sqrt{4 - x^2})}{[1 + \cos(\alpha x)]}$

$= 1^2 \cdot \frac{\alpha^2}{2} \cdot \frac{4}{2}$ (05)

$= \alpha^2$ (05)

$\because \alpha^2 = 16 \Rightarrow \alpha = 4 \ (\because \alpha > 0)$ [25]

(05)

Aliter

$\lim_{x \to 0} \frac{1 - \cos(\alpha x)}{\sqrt{4 + x^2} - \sqrt{4 - x^2}}$

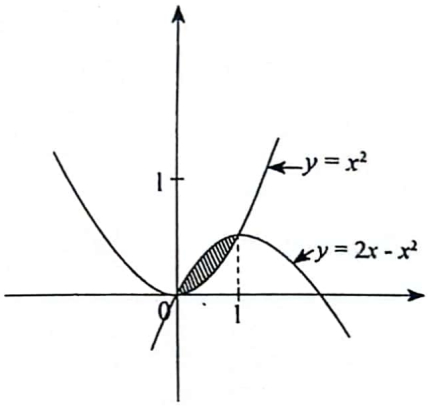
$= \lim_{x \to 0} \frac{2 \sin^2\left(\frac{\alpha x}{2}\right)}{\sqrt{4 + x^2} - \sqrt{4 - x^2}} \times \frac{(\sqrt{4 + x^2} + \sqrt{4 - x^2})}{(\sqrt{4 + x^2} + \sqrt{4 - x^2})}$ (05) (05)

$= \lim_{x \to 0} \left[\frac{\sin^2\left(\frac{\alpha x}{2}\right)}{\frac{\alpha x}{2}}\right]^2 \times \frac{\alpha^2}{4} \times \left(\sqrt{4 + x^2} + \sqrt{4 - x^2}\right)$ (05)

$= 1^2 \cdot \frac{\alpha^2}{4} \cdot 4 = \alpha^2$ (05)

$\alpha^2 = 16 \Rightarrow \alpha = 4 \ (\because \alpha > 0)$ (05) [25]

(06)



For points of intersection : $x^2 = 2x - x^2$

$x(x - 1) = 0$

$x = 0$ or $x = 1$

The required area $= \int_0^1 [(2x - x^2) - x^2] dx$ (05) (05) (05)

$= 2\int_0^1 (x - x^2)\, dx$

$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$ (05)

$= 2\left(1/2 - 1/3\right)$ (05)

$= \frac{1}{3}$ square units. [25]

(07) $\frac{dx}{d\theta} = 6 \sin \theta/2 \cos \theta/2 \times 1/2$ $\frac{dy}{d\theta} = 3 \sin^2\theta\cos\theta$ (05)

$= 3 \sin \theta/2 \cos \theta/2$ (05)

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{3 \sin^2 \theta \cdot \cos \theta}{3 \sin \theta/2 \cos \theta/2}$ (05)

$= 2 \sin \theta \cos \theta$

$\frac{dy}{dx} = \sin 2\theta$

$\left.\frac{dy}{dx}\right|_p = \frac{\sqrt{3}}{2}$

$\sin 2\theta = \frac{\sqrt{3}}{2} = \sin \pi/3$ (05)

$2\theta = \pi/3 \ (\because 0 < 2\theta < \pi/2)$

$\theta = \pi/6$ (05) [25]

(08) $l : 2x + 3y - k + \lambda (x - y + 1) = 0$ for some $\lambda \in \mathbb{R}$

Since l passes through the origin, $-k + \lambda = 0$ (05)

$\therefore \lambda = k$ (05)

$\therefore$ The equation of l is $(2 + k)x + (3 - k)y = 0$ (05)

(1 , 1) and (3 , 4) are on the same side

$\Rightarrow [(2 + k) + (3 - k)][3(2 + k) + 4(3 - k)] > 0$ (05)

$\Rightarrow 5(18 - k) > 0$

$k < 18$ (05) [25]

Aliter

$l : x - y + 1 + \lambda (2x + 3y - k) = 0$ for some $\lambda \in \mathbb{R}$ (05)

Since l passes through the origin

$1 - \lambda k = 0$

$\Rightarrow \lambda k = 1$

$\Rightarrow \lambda = 1/k \ (\because k \neq 0)$ (05)

$\therefore$ The equation of l is $\left(1 + \frac{2}{k}\right)x + \left(\frac{3}{k} - 1\right)y = 0$ (05)

(1 , 1) and (3 , 4) are on the same side

$\Rightarrow \left[1 + \frac{2}{k} + \frac{3}{k} - 1\right]\left[3 + \frac{6}{k} + \frac{12}{k} - 4\right] > 0$ (05)

$\Rightarrow \frac{5(18 - k)}{k^2} > 0 \Rightarrow k < 18 \ (\because k \neq 0)$ (05) [25]

(09) I. $\frac{(y - 2)(y - 4)}{(x - 1)(x + 5)} = -1$ for $x \neq 1, -5$ (05)

S: $(x - 1)(x + 5) + (y - 2)(y - 4) = 0$ (05)

$x^2 + y^2 + 4x - 6y + 3 = 0$

II. Let S^1 be the required circle.

Then $x^2 + y^2 - 2x - 2y + c^1 = 0$ (05)

S and S^1 intersects orthogonally

$\Rightarrow 2gg^1 + 2ff^1 = c + c^1$, where $g = 2, f = -3, g^1 = -1, f^1 = -1$ $c = 3$ and $c^1 = c^1$ (05)

$\Rightarrow 2.2 (-1) + 2 (-3) (-1) = 3 + c^1$

$c^1 = -1$ (05)

$S^1 : x^2 + y^2 - 2x - 2y - 1 = 0$ [25]

Aliter

I. S: $(x - 1)(x + 5) + (y - 2)(y - 4) = 0$ (10)

$x^2 + y^2 + 4x - 6y + 3 = 0$

II. Let S^1 be the required circle

Then $S^1 : (x - 1)^2 + (y - 1)^2 = r^2$ (05)

$x^2 + y^2 - 2x - 2y + 2 - r^2 = 0$

S and S^1 intersects orthogonally

$\Rightarrow 2gg^1 + 2ff^1 = c + c^1$, where

$g = 2\ f = -3, g^1 = -1, f^1 = -1, c = 3$ and $c^1 = 2 - r^2$

$2.2 (-1) + 2 (-3) (-1) = 3 + (2 - r^2)$ (05)

$\Rightarrow r^2 = 3$ (05)

$\therefore S^1 : x^2 + y^2 - 2x - 2y - 1 = 0$

(10) $\cos x + \cos 2x + \cos 3x = \sin x + \sin 2x + \sin 3x$ (05) (05)

$2 \cos 2x \cos x + \cos 2x = 2 \sin 2x \cos x + \sin 2x$

$\cos 2x (2 \cos x + 1) = \sin 2x (2 \cos x + 1)$ (05)

$\cos 2x = \sin 2x$ (05) $(\because 0 \leq x \leq \pi/2 \Rightarrow 2 \cos x + 1 \neq 0$

$\tan 2x = 1 \ (\because \cos 2x = 0 \Rightarrow \sin 2x \neq 0)$

$2x = \pi/4 \ (\because 0 < x < \pi/2)$

$x = \pi/8$ (05)

Part B

(11) (a) $f(x) = ax^2 + bx + c$

(a) $f(1) = a + b + c \neq 0$ (05)

$\therefore$ 1 is not a root of $f(x) =$ (05) [10]

$\alpha + \beta = -\frac{b}{a}$ (05) and $\alpha\beta = \frac{c}{a}$

$(\alpha - 1)(\beta - 1) = \alpha\beta - (\alpha + \beta) + 1$ (05) [15]

$= \frac{c}{a} + \frac{b}{a} + 1$

$= \frac{a + b + c}{a}$ (05)

Aliter :

$f(x) = a(x - \alpha)(x - \beta)$ (05)

$f(1) = a(1 - \alpha)(1 - \beta)$ (05)

$= a + b + c$

$(1 - \alpha)(1 - \beta) = \frac{a + b + c}{a}$ (05) [15]

Let $\alpha_1 = \frac{1}{\alpha - 1}$ and $\beta_1 = \frac{1}{\beta - 1}$

The quadratic equation with α_1 and β_1 as its roots is $(x - \alpha_1)(x - \beta_1) = 0$ (10)

i. e. $x^2 - (\alpha_1 + \beta_1)x + \alpha_1\beta_1 = 0$ ——— (1)

Now $\alpha_1 + \beta_1 = \frac{1}{\alpha - 1} + \frac{1}{\beta - 1} = \frac{\alpha + \beta - 2}{(\alpha - 1)(\beta - 1)}$ (05)

$= \frac{-b/a - 2}{\frac{(a + b + c)}{a}}$

$= \frac{-(2a + b)}{a + b + c}$ (05)

Also $\alpha_1\beta_1 = \frac{a}{a + b + c}$ (05)

By (1), the required equation is

$x^2 + \frac{(2a + b)}{a + b + c}x + \frac{a}{a + b + c} = 0$

$\Leftrightarrow (a + b + c)x^2 + (2a + b)x + a = 0$

$\Leftrightarrow g(x) = 0$, where $g(x) = (a + b + c)x^2 + (2a + b)x + a$ (05) [30]

$f(x) = ax^2 + bx + c$

$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$ (05)

$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$

$= a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}$ (05)

$\geqslant 0$ ($\because a > 0$)

$\geq -\frac{\Delta}{4a}$ and (= sign occurs when $x = -\frac{b}{2a}$) (05) (05)

$\therefore$ The minimum value of $f(x)$ is $-\frac{\Delta}{4a}$ (05)

i. e. $m_1 = -\frac{\Delta}{4a}$ [25]

Thus $m_2 = -\frac{\Delta}{4(a + b + c)}$ (05)

Where $\Delta = (2a + b)^2 - 4(a + b + c).a$ (05)

$= 4a^2 + 4ab + b^2 - 4a^2 - 4ab - 4ac$

$= b^2 - 4ac$

$= \Delta$ (05)

Hence $m_2 = \frac{-\Delta'}{4(a + b + c)}$

$= \frac{4am_1}{4(a + b + c)}$

$\Leftrightarrow (a + b + c)m_2 = am_1$ (05) [20]

$f(x) \geqslant 0$ for all $x \in \mathbb{R}$

$\Leftrightarrow m_1 \geqslant 0$ (05)

$\Leftrightarrow m_2 \geqslant 0$ $\quad \because m_2 = \frac{am_1}{(a + b + c)}$ (05)

$\Leftrightarrow g(x) \geqslant 0$ for all $x \in \mathbb{R}$ (05) [15]

(b) $P(1) = 5$, the remainder when $P(x)$ is divided by $(x - 1)$ (05)

$q(2) = 16$, the remainder when $q(x)$ is divided by $(x - 2)$ (05) [10]

$(x - 1)q(x) + 5 = (x - 1)(x^2 + 3x + 6) + 5$ (05)

$= x^3 + 3x^2 + 6x - x^2 - 3x - 6 + 5$

$= x^3 + 2x^2 + 3x - 1$ (05)

$= p(x)$ [10]

Note that $q(x) = (x - 2)(x + 5) + 16$ (05)

$p(x) = (x - 1)\{(x - 2)(x + 5) + 16\} + 5$

$= (x - 1)(x - 2)(x + 5) + 16x - 11$ (05)

Hence, the required remainder is $16x - 11$ (05) [15]

(12) (a) $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r x^r$, where ${}^nC_r = \frac{n!}{r!(n - r)!}$ (05) (05) [10]

For $r = 0, 1, 2, - - - - - n$

For $r = 0, 1, 2, - - - - n-1,$

$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{\frac{n!}{(r + 1)!(n - r - 1)!}}{\frac{n!}{r!(n - r)!}}$ (10)

$= \frac{\frac{1}{r + 1}}{\frac{1}{n - r}} = \frac{n - r}{r + 1}$ (05) ——— (1) [15]

Similarly, for $r = 0, 1, 2, - - - - n - 2$,

(1) $\Rightarrow \frac{{}^nC_{r+2}}{{}^nC_{r+1}} = \frac{n - r - 1}{r + 2}$ (05)

It is given that, ${}^nC_r : {}^nC_{r+1} : {}^nC_{r+2} = 1 : 2 : 3$ (05)

$\Rightarrow \frac{n - r}{r + 1} = 2$ and $\frac{n - r - 1}{r + 2} = \frac{3}{2}$ (05)

(05)

$\Rightarrow n - r = 2(r + 1)$ —(2) and $2(n - r - 1) = 3(r + 2)$

$\Rightarrow 4(r + 1) - 2 = 3r + 6$

$\Rightarrow r = 4$, and by (2) , $n = 14$ [25]

(05) (05)

(*b*) $\frac{10r + 9}{(2r - 3)(2r - 1)(2r + 1)} = \frac{r(Ar + B)}{(2r - 3)(2r - 1)} - \frac{(r + 1)(Ar + A + B)}{(2r - 1)(2r + 1)}$ (10)

$\Leftrightarrow 10r + 9 = r(Ar + B)(2r + 1) - (r + 1)(Ar + A + B)(2r - 3)$ (05)

$= r[2Ar^2 + (A + 2B)r + B] - (r + 1)[2Ar^2 + (2A + 2B - 3A)r - 3(A + B)]$

$= 2Ar^3 + (A + 2B)r^2 + Br - 2Ar^3 - (2B - A)r^2 + 3(A + B)r - 2Ar^2 - (2B - A)r + 3(A + B)$

$= -(4A + 2B)r + 3(A + B)$ for $r \in \mathbb{Z}^+$ (05)

$\Leftrightarrow r^1 : 4A + 2B = 10$ and $r^0 : 3A + 3B = 9$ (10)

$\Leftrightarrow A = 2$ and $B = 1$

(05) (05) [40]

So $U_r = g(r) - g(r + 1)$, where $g(r) = \frac{f(r)}{(2r - 3)(2r - 1)}$

with $f(r) = r(2r + 1)$

$r = 1$; $U_1 = g(1) - g(2)$

$r = 2$ $U_2 = g(2) - g(3)$ (10)

$r = n - 1$ $U_{n-1} = g(n - 1) - g(n)$

$r = n$ $U_n = g(n) - g(n + 1)$

$\sum_{r=1}^{n} U_r = g(1) - g(n + 1)$ (10)

$= \frac{1 \times 3}{(-1)(1)} - \frac{(n + 1)(2n + 3)}{(2n - 1)(2n + 1)}$ [30]

(05) (05)

$= -3 - \frac{(n + 1)(2n + 3)}{4n^2 - 1}$

$\lim_{n \to \alpha} \sum_{r=1}^{n} U_r = \lim_{n \to \alpha} \left[-3 - \frac{\left(1 + \frac{1}{n}\right)\left(2 + \frac{3}{n}\right)}{\left(4 - \frac{1}{n^2}\right)} \right]$ (05)

(05)

$= -3 - \frac{1}{2} = -\frac{7}{2}$ (05)

Hence $\sum_{r=1}^{\infty} U_r$ is convergent and the sum is $-\frac{7}{2}$ (05) [25]

(13) (*a*) $AX = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ (05)

$\lambda X = \begin{pmatrix} -\lambda \\ \lambda \end{pmatrix}$ (05)

Now $AX = \lambda X \Leftrightarrow \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\lambda \\ \lambda \end{pmatrix} = 2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Leftrightarrow \lambda = 2$ (05)

$AY = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\mu Y = \begin{pmatrix} -2\mu \\ \mu \end{pmatrix}$

Now $AY = \mu X \Leftrightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2\mu \\ \mu \end{pmatrix} = -1\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (05)

$\Leftrightarrow \mu = -1$ (05) [25]

$P = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$-a - 2c = 1 \Rightarrow c = -1$, $a = 1$

$-b - 2d = 0 \Rightarrow d = -1$, $b = 2$

$a + c = 0$

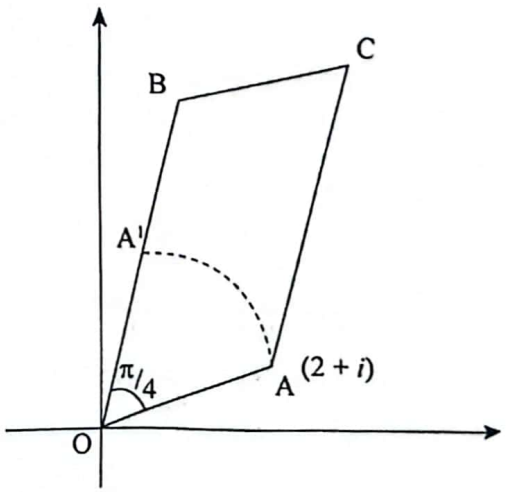
$b + d = 1$

$\therefore P^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ (15)

$AP = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$ (05)

$P^{-1}AP = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = D$ (05) [25]

(*b*)



The complex number represented by the point A^1 obtained by rotating OA about O counter clockwise, through an angle $\pi/4$ is

$z_1 = (2 + i)(\cos \pi/4 + i \sin \pi/4)$ (10)

$= \frac{1}{\sqrt{2}}(2 + i)(1 + i)$

$= \frac{1}{\sqrt{2}}(1 + 3i)$ (05)

$OA = OA' \Rightarrow OB = 2\,OA'$

The complex number z_2 represented by point B is given by

$z_2 = 2z_1$

$= \sqrt{2}(1 + 3i)$ (10) [25]

The complex number represented by point C.

$= (2 + i) + z_2$ (10)

$= 2 + i + \sqrt{2}(1 + 3i)$

$= (2 + \sqrt{2}) + (1 + 3\sqrt{2})\,i$ (05) [15]

(c) $w = \frac{2}{1 + i} + \frac{5\bar{z}}{2 + i}$

$= \frac{2(1 - i)}{2} + \frac{5\bar{z}(2 - i)}{5}$ (05)

$= 1 - i + \bar{z}(2 - i)$ (05)

$I_m\, w = -1 \Rightarrow -1 = -1 + I_m\, \bar{z}(2 - i)$

$\Rightarrow I_m\, \bar{z}(2 - i) = 0$

$\Rightarrow z(2 - i) = \bar{z}(2 + i)$ ——— ① (15)

$|w - 1 + i| = 5 \Rightarrow |z(2 - i)| = 5$

$\Rightarrow |z|\,|2 - i| = 5$

$\Rightarrow |z|\sqrt{5} = 5$ (15)

$|z| = \sqrt{5}$ ——— ②

① × z $\Rightarrow z^2(2 - i) = z\bar{z}(2 + i)$

② $\Rightarrow z\bar{z} = 5$

$\therefore z^2(2 - i) = 5(2 + i)$ (10)

$z^2 = \frac{2 + i}{2 - i} \cdot 5 = \frac{5}{5}(2 + i)^2$

$\therefore z = \pm(2 + i)$ (10) [60]

(14) (a) $f(x) = \frac{(x - 3)^2}{x^2 - 1}$ for $x \neq \pm 1$

$f'(x) = \frac{(x^2 - 1).2(x - 3) - 1(x - 3)^2.2x}{(x^2 - 1)^2}$ (15)

$= \frac{2(x - 3)[x^2 - 1 - x(x - 3)]}{(x^2 - 1)^2}$

$= \frac{2(x - 3)(3x - 1)}{(x^2 - 1)^2}$ (05) [20]

Horizontal Asymptote : $\lim_{x \to \pm\alpha} f(x) = 1$, Hence, it is $y = 1$ (05)

Note that

$\lim_{x \to 1^-} f(x) = -\infty$ and $\lim_{x \to 1^+} f(x) = \infty$

$\lim_{x \to -1^-} f(x) = \infty$ and $\lim_{x \to -1^+} f(x) = -\infty$ (05) [10]

Solve : $y = f(x)$ and $y = 1$ simultaneously.

i.e $\frac{(x - 3)^2}{x^2 - 1} = 1$ (05)

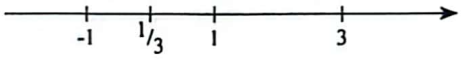
$\Leftrightarrow x^2 - 6x + 9 = x^2 - 1$

$\Leftrightarrow 6x = 10$

$\Leftrightarrow x = 5/3$ (05)

Hence, the required point $= (5/3, 1)$ (05) [15]

$f'(x) = 0 \Leftrightarrow x = 3$ or $x = 1/3$



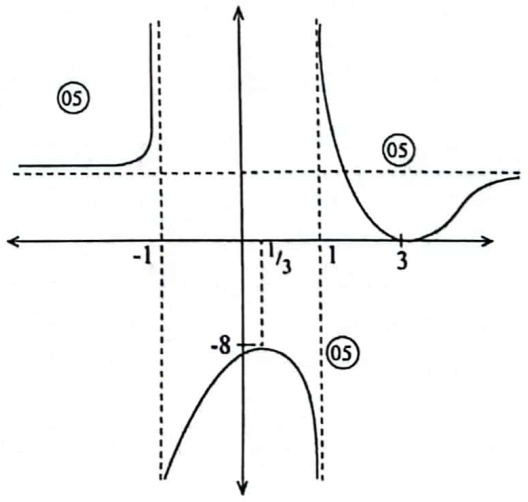
	$-\infty < x < -1$	$-1 < x < 1/3$	$1/3 < x < 1$	$1 < x < 3$	$3 < x < \infty$
sign of $f'(x)$	(+)	(+)	(-)	(-)	(+)
	(05)	(05)	(05)	(05)	(05)

There are two turning points :

$(1/3, -8)$ - Local max. (05)

$f(1/3) = \frac{(1/3 - 3)^2}{1/9 - 1} = \frac{64}{-8}$

(3 , 0) - Local min. (05) $= -8$



[50]

(*b*) $S = 2\pi(5r)h + \pi(5r^2) \times 2 - \pi r^2$ (10)

$= 10\pi rh + 49\pi r^2$ ——— (1)

$245\pi = \pi(5r)^2 \times h$

$\therefore h = \frac{245}{25r^2} = \frac{49}{5r^2}$ (05)

(1) $\Rightarrow S = 10\pi r \times \frac{49}{5r^2} + 49\pi r^2$

$= 49\pi\left(\frac{2}{r} + r^2\right); r > 0$ (05) [20]

$\frac{ds}{dr} = 49\pi\left(2r - \frac{2}{r^2}\right)$ (10)

(05) $\frac{ds}{dr} = 0 \Leftrightarrow 2r = \frac{2}{r^2} \Leftrightarrow r = 1$, (since $r > 0$) (05)

(05) $\frac{ds}{dr} < 0$ for $0 < r < 1$ and $\frac{ds}{dr} > 0$ for $r > 1$ (05)

$\therefore$ S is minimum when $r = 1$ (05) [35]

(15) (*a*) (i) $\int \frac{dx}{\sqrt{3 + 2x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-1)^2}}$ (10)

$= \sin^{-1}\left(\frac{x-1}{2}\right) + C_1$ where C_1 is an arbitary constant. [20] (10)

(ii) $\frac{d(\sqrt{3 + 2x - x^2})}{dr} = \frac{1}{2}\left(3 + 2x - x^2\right)^{-\frac{1}{2}} \times (2 - 2x)$ (10)

$= \frac{1 - x}{\sqrt{3 + 2x - x^2}}$

Hence $\int \frac{x-1}{\sqrt{3 + 2x - x^2}} dx = -\sqrt{3 + 2x - x^2} + C_2$, where C_2 is an arbitrary constant. (10)

$\int \frac{x+1}{\sqrt{3 + 2x - x^2}} dx = \int \frac{x-1}{\sqrt{3 + 2x - x^2}} dx + 2\int \frac{dx}{\sqrt{3 + 2x - x^2}}$ (10)

$= -\sqrt{3 + 2x - x^2} + 2\sin^{-1}\left(\frac{x-1}{2}\right) + C_3$

(05) (05) [40]

Where C_3 is an arbitrary constant.

(*b*) $\frac{2x - 1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2+1}$ (10)

$2x - 1 = A(x^2 + 1) + (Bx + C)(x + 1)$

$x^2 : 0 = A + B$

$x^1 : 2 = B + C$ $\quad A - C = -2$

$x^0 : -1 = A + C$

$A = -\frac{3}{2}$, $C = \frac{1}{2}$, $B = \frac{3}{2}$ (10)

$\frac{2x - 1}{(x+1)(x^2+1)} = \left(\frac{-3}{2}\right)\frac{1}{x+1} + \frac{1}{2}\frac{(3x+1)}{x^2+1}$ (05)

$\int \frac{2x - 1}{(x+1)(x^2+1)} dx$

$\left(= -\frac{3}{2}\ln|x+1| + \frac{3}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}(x) + C_4\right)$ (15)

Where C_4 is an arbitrary constant. [40]

(*c*) (i) $n \neq -1$

$\int x^n (\ln x)\, dx = \int \ln(x) \cdot \frac{d}{dx}\frac{x^{n+1}}{n+1} dx$. (10)

$= \left(\frac{x^{n+1}}{n+1}\right)\ln(x) - \int \frac{x^{n+1}}{(n+1)} \cdot \frac{1}{x} dx$ (10)

$= \left(\frac{x^{n+1}}{n+1}\right)\ln(x) - \frac{1}{(n+1)}\int x^n dx$

$= \left(\frac{x^{n+1}}{n+1}\right)\ln(x) - \frac{1}{(n+1)^2} x^{n+1} + C_5$ (10)

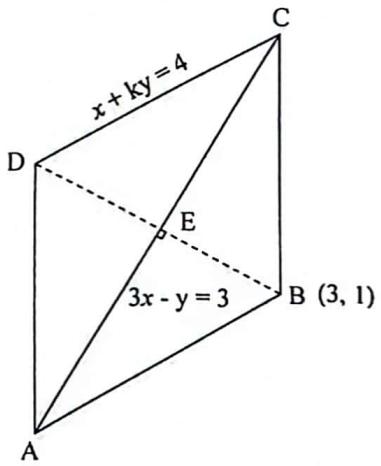
Where C_5 is an arbitrary constant. [30]

(ii) $\int_1^3 \frac{\ln x}{x} dx = \left[\frac{(\ln x)^2}{2}\right]_1^3$ (05) (15)

$= \frac{1}{2}((\ln 3)^2$ [20]

(05)

(16) (*a*)



(10)

The equation of BD : $y - 1 = -\frac{1}{3}(x - 3)$ ($\because$ BD$\perp$AC)

$\frac{x-3}{3} = \frac{y-1}{-1} = t$ (say) (10)

$\therefore x = 3 + 3t$ and $y = 1 - t$

Let t corresponds to the point D

Then $E \equiv \left(3 + \frac{3t}{2}, 1 - \frac{t}{2}\right)$ is on AC

$\therefore 3\left(3 + \frac{3t}{2}\right) - \left(1 - \frac{t}{2}\right) = 3$

$\Rightarrow 8 + 5t = 3 \Rightarrow t = -1$ (10)

$\therefore$ D $\equiv (0, 2)$ (10)

It is on DC

$O + k \times 2 = 4$ (10)

$k = 2$ [50]

$x + 2y = 4$, $3x - y = 3$ $\Rightarrow 7x = 10$; $7y = 9$

$C \equiv \left(\frac{10}{7}, \frac{9}{7}\right)$ (10)

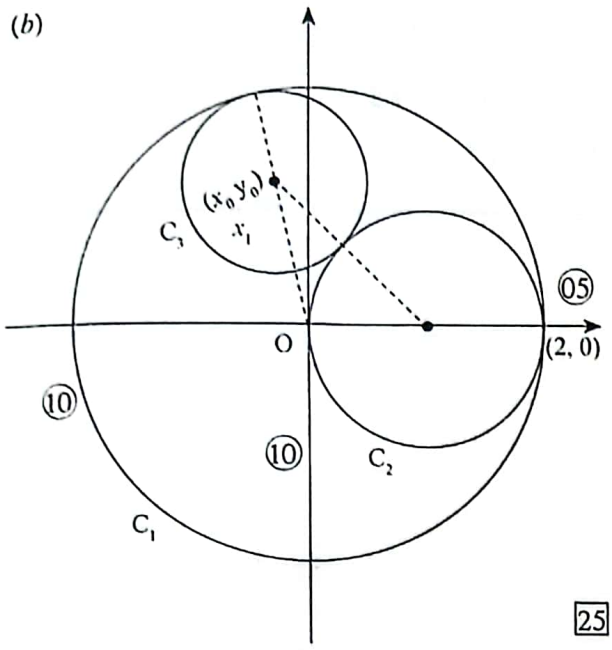
Equation of BC :

$y - 1 = \frac{2/7}{11/7}(x - 3)$

$-11y + 11 = 2x - 6$

$2x + 11y = 17$ (10) [20]

(*b*)



05 10 10 05 25

C_3 touches C_1 internally $\Rightarrow 2 - r_1 = \sqrt{x_0^2 + y_0^2}$ (15)

C_3 touches C_2 externally $\Rightarrow 1 + r_1 = \sqrt{(x_0 - 1)^2 + y_0^2}$ (15)

$3 = \sqrt{x_0^2 + y_0^2} + \sqrt{(x_0 - 1)^2 + y_0^2}$ (05)

$(x_0 - 1)^2 + y_0^2 = 9 - 6\sqrt{x_0^2 + y_0^2} + x_0^2 + y_0^2$

$x_0^2 - 2x_0 + 1 + y_0^2 = 9 - 6\sqrt{x_0^2 + y_0^2} + x_0^2 + y_0^2$ (10)

$2x_0 + 8 = 6\sqrt{x_0^2 + y_0^2}$ (05)

$\Rightarrow (x_0^2 + y_0^2) = x_0^2 + 8x_0 + 16$

$\Rightarrow 8x_0^2 + 9y_0^2 - 8x_0 - 16 = 0$

Hence $(x_0 \ y_0)$ lies on the curve, $8x^2 + 9y^2 - 8x - 16 = 0$ (05) 55

(17) (*a*) $\tan(\alpha + \beta) = \dfrac{\tan \propto + \tan \beta}{1 - \tan \propto \tan \beta}$ (10) 10

Let $\alpha = \beta = \theta$ (05)

$\tan 2\theta = \dfrac{\tan\theta + \tan\theta}{1 - \tan\theta \ \tan\theta}$

$= \dfrac{2\tan\theta}{1 - \tan^2\theta}$ (05) 10

$\tan 3\theta = \tan(\theta + 2\theta)$ (05)

$= \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta}$ (05)

$= \dfrac{\tan\theta(1 - \tan^2\theta) + 2\tan\theta}{1 - \tan^2\theta - 2\tan^2\theta}$ (05)

$= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ (05) 20

$\theta = {}^{5\pi}/_{12} \Rightarrow \tan {}^{5\pi}/_{4} = \dfrac{3\tan {}^{5\pi}/_{12} - \tan^3 {}^{5\pi}/_{12}}{1 - 3\tan^2({}^{5\pi}/_{12})}$ (05) (05)

$\Rightarrow -3\tan({}^{5\pi}/_{12}) + \tan^3({}^{5\pi}/_{12}) - 3\tan^2({}^{5\pi}/_{12}) + 1 = 0$

(Since $\tan({}^{5\pi}/_{4}) = 1$) (05)

$\Rightarrow \tan({}^{5\pi}/_{12})$, is a solution of $x^3 - 3x^2 - 3x + 1 = 0$ (05) 15

$(x + 1)(x^2 - 4x + 1) = 0$

$\tan({}^{5\pi}/_{12}) \neq -1 \Rightarrow \tan({}^{5\pi}/_{12})$ is a solution of $x^2 - 4x + 1 = 0$ (05)

i.e (05) $x = \dfrac{4 \pm \sqrt{16 - 4}}{2} = \dfrac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$ (05)

${}^{5\pi}/_{12} > {}^{\pi}/_{4} \Rightarrow \tan({}^{5\pi}/_{12}) > 1$ (05)

Since $2 - \sqrt{3} < 1$, $\tan({}^{5\pi}/_{12}) = 2 + \sqrt{3}$ (05) 25

(*b*) $O < A < \pi$

$\dfrac{1 - \cos A}{1 + \cos A} = \dfrac{2\sin^2({}^{A}/_{2})}{2\cos^2({}^{A}/_{2})}$ (05) $= \tan^2({}^{A}/_{2})$ (05) (05) 15

The Cosine Rule says that

$a^2 = b^2 + c^2 - 2bc\cos A$ (05)

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Now $\tan^2({}^{A}/_{2}) = \dfrac{1 + \dfrac{a^2 - b^2 - c^2}{2bc}}{1 - \dfrac{(a^2 - b^2 - c^2)}{2bc}}$ (10)

$= \dfrac{2bc + a^2 - b^2 - c^2}{2bc - a^2 + b^2 + c^2}$

$= \dfrac{(a - b + c)(a + b - c)}{(b + c - a)(a + b + c)}$ (05) (05)

$\Rightarrow (a + b + c)(b + c - a)\tan^2({}^{A}/_{2}) = (a + b - c)(a + c - b)$ (05) 30

(*c*) Let $\alpha = \sin^{-1}({}^{3}/_{5})$ and $\beta = \sin^{-1}({}^{5}/_{13})$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ (05)

$= \dfrac{3}{5}\sqrt{1 - \dfrac{25}{169}} + \sqrt{1 - \dfrac{9}{25}} \cdot \dfrac{5}{13}$ (05)

$= \dfrac{3}{5} \cdot \dfrac{12}{13} + \dfrac{4}{5} \cdot \dfrac{5}{13}$

$= \dfrac{56}{65}$ (05)

Since ${}^{3}/_{5} < \sqrt{3}/_{2}$, We have $0 < \alpha < {}^{\pi}/_{3}$

Similarly since ${}^{5}/_{13} < {}^{1}/_{2}$ We have $0 < \alpha < {}^{\pi}/_{6}$

$\therefore 0 < \alpha + \beta < {}^{\pi}/_{2}$ and hence, $\alpha + \beta = \sin^{-1}({}^{56}/_{65})$ (10) 25