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G.C.E. (Advanced Level) Examination - August 2015
Combined Mathematics - I
Scheme of Marking

Part A

(01) If $n = 1$, $8^n - 3^n = 8 - 3 = 5$ Result is true for $n = 1$ (05)

Suppose the result is true for $n = p$.

$\rightarrow 8^p - 3^p = 5m$, Where m is an integer (05)

Consider $n = p + 1$; $8^{p+1} - 3^{p+1} = 8^p (5+3) - 3^{p+1}$

$= (5m+3^p)(5+3) - 3^{p+1}$ (05)

$= 8\times5m+5\times3^p+3^{p+1} - 3^{m+1}$

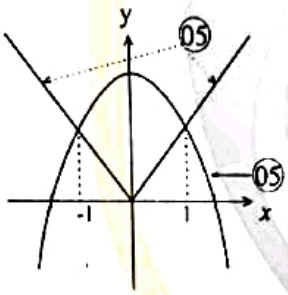
$= 5(8m+3^p)$ (05) $8m+3^p \in Z^+$

Hence, if the result is true for $n = p$, then it is also true for $n = p + 1$

Thus by the principle of Mathematical Induction, the result is true for all $n \in Z^+$ (05)

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(02) $|x| < 2 - x^2$



(05) (05)

Aliter

for $x \geq 0$

$x = 2 - x^2$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = 1$, (05)

for $x < 0$

$-x = 2 - x^2$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = -1$ (05)

solution $\{x | -1 < x < 1\}$ (05)

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Aliter

for $x \geq 0$

$x < 2 - x^2$ (05)

$x^2 + x - 2 < 0$

$(x - 1)(x + 2) < 0$

$\Rightarrow 0 \leq x < 1$ (05)

for $x < 0$

$-x < 2 - x^2$ (05)

$x^2 - x - 2 < 0$

$(x + 1)(x - 2) < 0$

$\Rightarrow -1 < x < 0$ (05)

Solution $\{x | -1 < x < 1\}$ (05)

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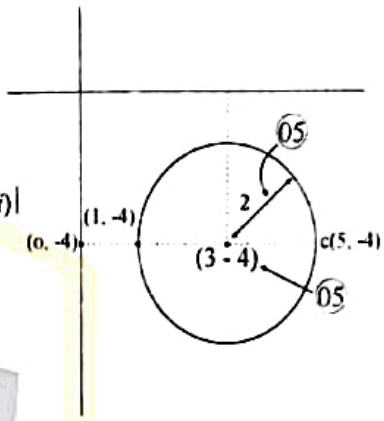
(03) C is the circle with center (3, -4) and radius = 2

(05)

$|z - 3 + 4i| = 2$

$|z - (3-4i)| = 2$

$|z + 4i| = |z - (-4i)|$



(05) (05)

For Z on C the greatest value of $|z + 4i|$ is 5; the smallest value of $|2 + 4i|$ is 1 (05)

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(04) $\left(3x + \frac{2}{x}\right)^n - \sum_{r=0}^{n} {}^nC_r (3x)^{n-r} \left(\frac{2}{x}\right)^r$

$\sum_{r=0}^{n} {}^nC_r \, 3^{n-r} \cdot 2^r \cdot x^{n-2r}$ (05)

$n - 10 = n - 2r \Rightarrow r = 5$ (05)

So the coefficient of x^{n-10} is ${}^nC_5 \, 3^{n-5} \cdot 2^5$

${}^nC_5 \, 3^{n-5} \times 32 < 100 \Rightarrow 3^{n-5} < \frac{100}{32}$, $\because {}^nC_5 > 1$ (05)

Given that $n \geq 5$ $n = 5$ or $n = 6$ are the possible values.

$n = 5$ $3^0 < \frac{100}{32}$ valid.

$n = 6$ $6 \times 3 < \frac{100}{32}$ not valid

$\therefore n = 5$ (05) } (05)

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(05) $\lim_{x\to 0} \dfrac{(x+\sqrt{2})^4 - 4}{\sin 4x} = \lim_{x\to 0} \left[\dfrac{(x+\sqrt{2})^4 - (\sqrt{2})^4}{(x+\sqrt{2}-\sqrt{2})} \cdot \dfrac{1}{4\,\frac{\sin 4x}{4x}} \right]$ (05) (05)

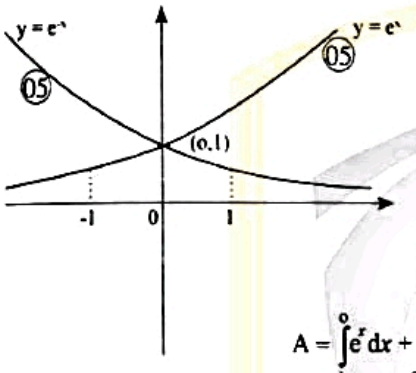
$= \dfrac{1}{4} \lim_{x\to 0} \dfrac{(x+\sqrt{2})^4 - (\sqrt{2})^4}{(x+\sqrt{2}-\sqrt{2})} \cdot \dfrac{1}{\lim_{x\to 0} \frac{\sin 4x}{4x}}$ (05) (05)

$= \dfrac{1}{4} \cdot 4(\sqrt{2})^3 \cdot \dfrac{1}{1}$ (using the given result) (05)

$= (\sqrt{2})^{3+1} = 2\sqrt{2}$.

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(06)



(05) (05)

$A = \int_{-1}^{0} e^x dx + \int_{0}^{1} e^{-x} dx$ (05)

$= [e^x]_{-1}^{0} + [-e^{-x}]_{0}^{1}$ (05)

$= 1 - e^{-1} - e^{-1} + 1$

$= 2 - 2e^{-1}$ (05)

$= 2\left(1 - \frac{1}{e}\right)$

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(07) Parametric equations of curve C :

$x = 2 + \cos 2\theta$, $y = 4\sin\theta$

$\dfrac{dx}{d\theta} = -2\sin 2\theta$, $\dfrac{dy}{d\theta} = 4\cos\theta$ (05)

$\dfrac{dy}{dx} = \dfrac{4\cos\theta}{-4\sin\theta\cos\theta} = -\dfrac{1}{\sin\theta}$ (05)

When $\theta = \pi/4$, $\dfrac{dy}{dx} = -\sqrt{2}$

Gradient of normal $= \dfrac{1}{\sqrt{2}}$ (05)

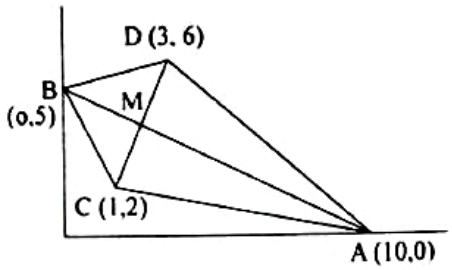
equation of normal at point $(2, 2\sqrt{2})$ (05)

$y - 2\sqrt{2} = \dfrac{1}{\sqrt{2}}(x - 2)$ (05)

$\sqrt{2}y - 4 = x - 2 \Rightarrow x - \sqrt{2}y + 2 = 0$

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(08)



Midpoint of CD is (2,4)

Equation of line AB : $\dfrac{y-5}{x-0} = 1/2$

$\Rightarrow x + 2y - 10 = 0$

(2, 4) satisfies $x + 2y - 10 = 0$ ($\because 2 + 2\times 4 - 10 = 0$) (05)

Also gradient of CD $= \dfrac{6-2}{3-1} = \dfrac{4}{2} = 2$ $\therefore$ CD is

Perpendicular to AB (05)

Area of ACBD $= \dfrac{1}{2}$ AB (MD + MC) $= \dfrac{1}{2}$ AB.CD

$= \dfrac{1}{2}\sqrt{100+25}\ \sqrt{2^2+4^2}$ (05)

$= 25$

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(09) Circle $x^2 + y^2 - 2x - 2y + 1 + \lambda(y-1) = 0$,

where λ is a parameter (05)

Passes through the origin $0 \Rightarrow 1 - \lambda = 0$, $\lambda = 1$ (05)

Required equation of circle $x^2 + y^2 - 2x - y = 0$ (05)

$(x-1)^2 + (y - 1/2)^2 = 1 + \dfrac{1}{4} = (\frac{\sqrt{5}}{2})^2$

centre $(1, 1/2)$, radius $= \frac{\sqrt{5}}{2}$

(05) (05)

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(10) Both α and β are acute angles.

$\sin\alpha + \sin\beta = 1$, $2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) = 1$ (05)

$\cos\alpha + \cos\beta = \sqrt{3}$ $2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) = \sqrt{3}$ (05)

Dividing, $\tan\left(\frac{\alpha+\beta}{2}\right) = 1/\sqrt{3}$. (05)

Since $0 < \dfrac{\alpha+\beta}{2} < \pi/2$, $\dfrac{\alpha+\beta}{2} = \pi/6 \Rightarrow \alpha + \beta = \pi/3$

(05) (05)

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Part B

(11) (a) $F(x) = (x^2 - ax + 1)(x^2 - \beta x + 1)$

$= x^4 - (x + \beta)x^3 + (2 + \alpha\beta)x^2 - (\alpha + \beta)x + 1$ (05)

(i) If $F(x) = 0$ and $G(x) = 0$ have the same roots,

then $G(x) = 6F(x) \Leftrightarrow 6x^4 - 35x^3 + 62x^2 - 35x + 6$

$= 6[x^4 - (\alpha+\beta)x^3 + (2+\alpha\beta)x^2 - (\alpha+\beta)x + 1]$ (05)

Equating the coefficients : $\alpha + \beta = 35/6$ (05)

$2 + \alpha\beta = \frac{62}{6} \Rightarrow \alpha\beta = \frac{62}{6} - 2 = \frac{50}{6}$ (05)

The quadratic equation whose roots are α and β is given by $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - \frac{35}{6}x + \frac{50}{6} = 0$ (05)

$\Rightarrow 6x^2 - 35x + 50 = 0$ [25]

$\Rightarrow (3x - 10)(2x - 5) = 0$ (05)

$\Rightarrow x = 10/3$ or $x = 5/2$

Take $\alpha = 10/3$ and $\beta = 5/2$

(05) (05)

Roots of the equation $G(x) = 0$ are given by $F(x) = 0$

$\Leftrightarrow \left(x^2 - \frac{10}{3}x + 1\right)\left(x^2 - \frac{5}{2}x + 1\right) = 0$ (05)

$\Leftrightarrow (3x^2 - 10x + 3)(2x^2 - 5x + 2) = 0$

$\Leftrightarrow (x - 3)(3x - 1)(x - 2)(2x - 1) = 0$ (05)

$\Leftrightarrow x = 2, 1/2, 3$ or $1/3$

(05) (05) [35]

(ii) If $H(x) \equiv F(x)$,

$x^4 + x^2 + 1 = x^4 - (\alpha+\beta)x^3 + (2+\alpha\beta)x^2 - (\alpha+\beta)x + 1$

$\Leftrightarrow \alpha + \beta = 0$ (05)

$2 + \alpha\beta = 1 \Rightarrow \alpha\beta = -1$ (05)

$\Rightarrow \alpha(-\alpha) = -1$ *i. e.* $\alpha^2 = 1$

$\Rightarrow \alpha = \pm 1$

so $\beta = \mp 1$ } (05)

Aliter

$\therefore \alpha$ and β are the roots of the equation $x^2 - 1 = 0$

$\Rightarrow x = \pm 1$ (05)

take $\alpha = 1$ and $\beta = -1$

$H(x) = 0$

$\Leftrightarrow F(x) = 0 \Rightarrow (x^2 - x + 1)(x^2 + x + 1) = 0$ (05)

$\Leftrightarrow x^2 - x + 1 = 0$ or $x^2 + x + 1 = 0$

$\Leftrightarrow \Delta = (-1)^2 - 4(1)(1) < 0$ [25]

$H(x) = 0$ has no real roots (05)

(b) (i) $F(x) = 2x^4 + \gamma x^3 + \delta x + 1$

since $f(-1/2) = 0$

$2(1/16) + \gamma(-1/8) + \delta(-1/2) + 1 = 0$

$\Rightarrow 1 - \gamma - 4\delta + 8 = 0$

$\Rightarrow \gamma + 4\delta = 9$ (05)

since $f(-2) = 21$

$2(16) + \gamma(-8) + \delta(-2) + 1 = 21$

$\Rightarrow 8\gamma + 2\delta = 12$

$\Rightarrow 4\gamma + \delta = 6$ (05)

By solving , $15\gamma = 15 \Rightarrow \gamma = 1$ and $\delta = 2$ (05)

$\therefore F(x) = 2x^4 + x^3 + 2x + 1$ (05)

$= (2x + 1)(x^3 + 1)$, $\because f(-1/2) = 0$

$= (2x + 1)(x + 1)(x^2 - x + 1)$

so the two real linear factors of $f(x)$ are $x + 1$ (05)

and $2x + 1$. (05) [30]

(ii) $(x^2 + x + 1)P(x) + (x^2 - 1)Q(x) = 3x$

Let $P(x) = ax + b$ and $Q(x) = cx + d$

Then $(x^2 + x + 1)(ax + b) + (x^2 - 1)(cx + d) = 3x$ (05)

By equating the coefficients,

$a + c = 0$ — (1) (05)

$b + a + d = 0$ — (2) (05)

$b + a - c = 3$ — (3) (05)

$b - d = 0$ — (4) (05)

(1) + (3) $\Rightarrow 2a + b = 3$ — (5)

(2) + (4) $\Rightarrow 2b + a = 0$ — (6)

By (5) and (6) $a = 2$ and $b = -1$

By (1), $c = -2$, also by (4) $d = -1$

$\therefore P(x) = 2x - 1$ and $Q(x) = -2x - 1$

(05) (05) [35]

Aliter

(ii) $(x^2 + x + 1)P(x) + (x^2 - 1)Q(x) = 3x$

Let $P(x) = ax + b$ and $Q(x) = cx + d$

Then $(x^2 + x + 1)(ax + b) + (x^2 - 1)(cx + d) = 3x$ (05)

$x = 1$: $3(a + b) = 3 \Rightarrow a + b = 1$ (05)

$x = -1$: $-a + b = -3$ (05)

$x = 0$: $-1 - d = 0$ (05)

$x = 1/2$: $(\frac{1}{4} - 1)(-c/2 - 1) = 3/2$ (05)

Then $a = 2$, $b = -1$, $c = -2$, $d = -1$

$P(x) = 2x - 1$ $Q(x) = -2x - 1$

(05) (05) [35]

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(12) (a)

Sportsmen sportswomen		Male singers (MS)	Female singers (FS)	Actors	Actresses
2	3	5	6	4	2
Panel : Head		three others			

(i) 1 Head + 1FS + 1MS + 1 other $\Rightarrow {}^3C_1 \times {}^6C_1 \times {}^5C_1 \times {}^4C_1$ 05

1 Head + 2FS + 1MS $\Rightarrow {}^3C_1 \times {}^6C_2 \times {}^5C_1$ 05 } - 05

1 Head + 1FS + 2MS $\Rightarrow {}^3C_1 \times {}^6C_1 \times {}^5C_2$ 05 }

$= 900 - 375 - 300 = 1575$ 05

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(ii) Female Head + 1 Woman + 2 Men

05 05

$\Rightarrow {}^3C_1 \times {}^5C_1 \times {}^4C_2$ - 05

Male Head + 2 Women + 1 Man

$\Rightarrow {}^3C_1 \times {}^5C_2 \times {}^4C_1$

05 05

$= 864 + 504$

$= 1368$ 05 30

(iii) Sport woman as Head – any other 3

05 05

$\Rightarrow {}^3C_1 \times {}^7C_3$

$= 2040$ 05 15

(b) $A(r+5)^2 - B(r+1)^2 \equiv r + C$

$A(r^2 + 10r + 25) - B(r^2 + 2r + 1) \equiv r + C$

Equating coefficients:

$r^2 : A - B = 0$ 05

$r : 10A - 2B = 1$ 05

$r^0 : 25A - B = C$ 05

$A = B = 1/8$, $C = 24A = 3$

05 05

So $(r+5)^2 - (r+1)^2 \equiv 8(r+3)$

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Consider given u_r ;

$u_r = \frac{8(r+3)}{(r+1)^2(r+3)^2(r+5)^2}$ 05 $= \frac{(r+5)^2 - (r+1)^2}{(r+1)^2(r+3)^2(r+5)^2}$ 05

$= \frac{1}{(r+1)^2(r+3)^2} - \frac{1}{(r+3)^2(r+5)^2}$

$f(r) - f(r+2)$, where $f(r) = \frac{1}{(r+1)^2(r+3)^2}$ 05

15

$u_r = f(r) - f(r+2)$

For r = 1, 2, .. n

$u_1 = f(1) - f(3)$

$u_2 = f(2) - f(4)$ 10

$u_3 = f(3) - f(5)$

$u_{n-2} = f(n-2) - f(n)$

$u_{n-1} = f(n-1) - f(n-1)$

$u_n = f(n) - f(n+2)$ 10

$\sum_{r=1}^{n} u_r = f(1) + f(2) - f(n+1) - f(n+2)$ 05

$= \frac{1}{2^2 4^2} + \frac{1}{3^2 5^2}$ 05 $- \frac{1}{(n+2)^2(n+4)^2} \quad \frac{1}{(n+3)^2(n+5)^2}$ 05

$\because \sum_{r=1}^{\infty} u_r = \frac{1}{8^2} + \frac{1}{15^2}$,

$\because$ last two terms tend to zero as $n \to \infty$ 05 40

(13) (a) $A = \begin{pmatrix} 0 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} a & b & o \\ c & d & o \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 4 \\ 2 & 3 \\ 1 & 2 \end{pmatrix}$

(i) $AC = \begin{pmatrix} 4 & -3 & 0 \\ 0 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ 05

$CA = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 05

10

(ii) $BC = \begin{pmatrix} 3a + 2b & 4a + 3b \\ 3c + 2d & 4c + 3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 05

1st row

$3a + 2b = 1$ 05 $\quad 3a + 2\left(\frac{-4a}{3}\right) = 1$

$4a + 3b = 0$ $\quad a = 3,\ b = -4$ 05

05

2nd row

$3c + 2d = 0$ 05 $\quad 4c + 3\left(\frac{-3c}{2}\right) = 1$

$4c + 3d = 1$ $\quad c = -2,\ d = 3$

05 05

$\because A = \begin{pmatrix} 3 & -4 & 0 \\ -2 & 3 & 0 \end{pmatrix}$

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(iii) $(\lambda A + \mu B)\, C = \lambda AC + \mu BC = (\lambda + \mu)\, I_2 = I_2$ (05)

$\Rightarrow (\lambda + \mu - 1)\, I_2 = 0 \Rightarrow \lambda + \mu = 1$ (05) [10]

Let $D = \begin{pmatrix} -3 & 8 & -6 \\ 2 & -5 & 4 \end{pmatrix} = \alpha \begin{pmatrix} 0 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} 3 & -4 & 0 \\ -2 & 3 & 0 \end{pmatrix}$ (05)

1st column	3rd column
$\alpha = -1$ (05)	$\beta = 2$ (05) [15]

Hence $D = 2A - B$ and $DC = (2A - B)\, C = 2AC - BC$ (05)

$= 2I_2 - I_2 = I_2$ (05)

[10]

(b) $z = \cos\theta + i \sin\theta \quad (-\pi < \theta \leq \pi)$

$|z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$;

Point representing z lies on the circle C of radius 1 and centre O. (05)

(05)

$\bar{z} = \cos\theta - i \sin\theta = \frac{1}{z}$ (05)

$z + \bar{z} = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ (05)

$z - \bar{z} = 2i \sin\theta \Rightarrow \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ (05) [25]

(i) $w = \frac{2z}{z^2+1} = \frac{2}{z + \frac{1}{z}} = \frac{1}{\cos\theta} = \sec\theta$ (05) $\therefore I_m(w) = 0$

$t = \frac{z^2 - 1}{z^2 + 1} = \frac{z - \frac{1}{z}}{z + \frac{1}{z}} = \frac{i \sin\theta}{\cos\theta} = i \tan\theta$; $\therefore$ Re (t) = 0 (05) [10]

$w^2 + t^2 = \sec^2\theta + (i \tan\theta)^2 = \sec^2\theta - \tan^2\theta = 1$ (05)

[05]

(ii) $w = 2 \Rightarrow \frac{1}{\cos\theta} = 2$ or $\cos\theta = 1/2$ (05)

In the given range

$\theta = \pm \pi/3 \quad z = \frac{1}{z} + i\frac{\sqrt{3}}{2}, \; z = \frac{1}{z} - i\frac{\sqrt{3}}{2}$

(05) (05) [15]

(iii) $t = i \Rightarrow i \tan\theta = i \Rightarrow \tan\theta = 1$ (05)

In the given range $\theta = \pi/4$ or $\theta = (-3\pi/4)$ (05)

$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, i$ or $z = -\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ (05) [15]

(14) (a) $y = x \sin(1/x), \quad x \neq 0$

(i) $\frac{dy}{dx} = \sin(1/x) + x(-1/x^2)\cos(1/x)$ (05)

Multiplying by x throughout. (05)

$\Rightarrow x\frac{dy}{dx} = y - \cos(1/x)$ [10]

(ii) Differentiating w. r. t. x. and using $\sin(1/x) = y/x$

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{dx} + \sin(1/x)(-1/x^2)$ [10]

Multiplying by

$x^3 \Rightarrow x^4 \frac{d^2y}{dx^2} + y = 0$ (05) [15]

(b) $f(x) = \frac{2x^2 + 1}{(x-1)^2}, \quad x \neq 1$

$f'(x) = \frac{(x-1)^2 \cdot 4x - (2x^2+1) \cdot 2(x-1)}{(x-1)^4}$ (10)

$= \frac{(x-1)\,4x - 2(2x^2+1)}{(x-1)^3}$

$= -2\frac{(2x+1)}{(x-1)^3}; x \neq 1$ (05) [15]

$f'(x) = 0$ when $x = -1/2$ (05) [05]

$f'(x)$ does not exist when $x = 1$

$\Rightarrow$ vertical asymptote at $x = 1$ (05)

	$x < (-1/2)$	$-1/2 < x < 1$	$1 < x$
sign of $f'(x)$	(-)	(+)	(-)
	\	/	\

[10]

$F(-1/2) = \frac{2(-1/2)^2 + 1}{(-1/2 - 1)^2} = \frac{3/2}{(-3/2)^2} = 3/2$

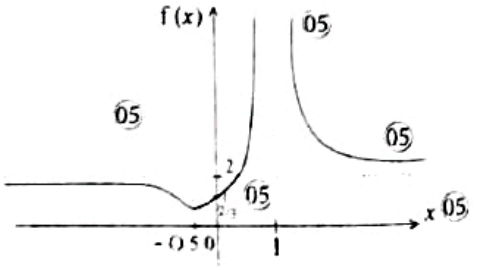
$\therefore F(x)$ takes local minimum at the point

$(-1/2, 2/3)$ (05)

$x > 1$ and $f'(x) < 0$

$x \to +\infty \quad f(x) \to 2$

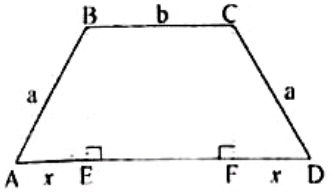
$x \to -\infty \quad f(x) \to 2$ (05)



[50]

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(c)



Area : $S(x) = 2 \times \frac{1}{2} x\ (\sqrt{a^2-x^2}) + b\sqrt{a^2-x^2} = (b+x)\sqrt{a^2-x^2}$

(10) **10**

Substituting $a = \sqrt{6}$ $b = 4$

$S(x)=(4+x)\sqrt{6-x^2}$ (5) $\Rightarrow \frac{ds}{dx} = (4+x)\frac{1}{2\sqrt{6-x^2}}(-2x)+\sqrt{6-x^2}$ (05)

$$\frac{ds}{dx} = \frac{-x(4+x)+6-x^2}{2\sqrt{6-x^2}}$$

$$\frac{ds}{dx} = \frac{-2x^2-4x+6}{\sqrt{6-x^2}} = \frac{-2(x^2+2x-3)}{\sqrt{6-x^2}}$$ (05)

$\frac{ds}{dx} = 0$ when $x^2 + 2x - 3 = 0$ (05)

$(x + 3)(x - 1) = 0$

Since x is positive $x = 1$ gives a turning point (05)

	$0 < x < 1$	$1 < x < \sqrt{6}$
sing of $S'(x)$	+	-
	(05)	(05)

$S(x)$ is maximum at $x = 1$ (05)

maximum value of S is

$S(1) = (4 + 1)\sqrt{6 - 1} = 5\sqrt{5}\ S_q$ units

(05) **45**

(15) (a) Let $y = \pi - x$

$$\int_0^{\pi} f(x)\,dx = \int_{\pi}^{0} f(\pi-y)\,(-dy) = \int_0^{\pi} f(\pi-y)\,dy = \int_0^{\pi} f(\pi-x)\,dx$$ (05) (05)

10

$$\int_0^{\pi/2} \sin^2 x\ dx = \frac{1}{2}\int_0^{\pi/2} (1-\cos 2x)\,dx = \frac{1}{2}[x]_0^{\pi/2} - 0 = \pi/4 .$$ (05) (05)

$\because [\sin 2x]_0^{\pi/2} = 0$ **10**

using the first result.

$$\int_0^{\pi} x\sin^2 x\,dx = \int_0^{\pi} (\pi - x)\sin^2(\pi - x)\,dx$$

$$= \pi\int_0^{\pi} \sin^2 x\,dx - \int_0^{\pi} x\sin^2 x\,dx$$ (05)

$$\therefore 2\int_0^{\pi} x\sin^2 x\,dx = \pi\left[\int_0^{\pi/2} \sin^2 x\,dx + \int_{\pi/2}^{\pi} \sin^2 x\,dx\right]$$ (05)

$$= \pi\left[\pi/4 + J\right] \text{ where } J = \int_{\pi/2}^{\pi} \sin^2 x\,dx$$ (05)

substituting $\pi - x = y$ (05)

$$J = \int_{\pi/2}^{0} \sin^2(\pi - y)(-dy) = \int_0^{\pi/2} \sin^2 y\,dy = \pi/4$$ (05)

$$\int_0^{\pi} x\sin^2 x\,dx = \pi/2\left[\pi/4 + \pi/4\right] = \frac{1}{2}(\pi \cdot \pi/2) = \frac{\pi^2}{4}.$$ (05)

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(b) Substitution :

$t = x^2 \Rightarrow dt = 2x\,dx$ (05)

$\therefore \int x^3 e^{x^2}\,dx = \frac{1}{2}\int t\,e^t\,dt$ (05)

$= \frac{1}{2}\int t\frac{d}{dt}(e^t)\,dt = \frac{1}{2}te^t - 1/2\int e^t\,dt$ (05)

$= \frac{1}{2}te^t - \frac{1}{2}e^t + c$ Substituting $t = x^2$, $\int x^3 e^{x^2}\,dx$

(05)

$\frac{1}{2}e^{x^2}(x^2 - 1) + c$

(05) (05)

30

(c) $\frac{1}{x^3-1} \equiv \frac{A}{x-1} + \frac{BX+C}{x^2+x+1}$

$1 \equiv A(x^2 + x + 1) + (x - 1)(Bx + C)$

Equating coefficients of x^0 : $1 = A - C$

Equating coefficients of x^1 : $0 = A + C - B$

Equating coefficients of x^2, $0 = A + B$

solving $A = \frac{1}{3}$ (05) $B = -\frac{1}{3}$ (05) $C = -\frac{2}{3}$ (05)

15

$$\int\frac{dx}{x^3-1} = \frac{1}{3}\int\frac{dx}{x-1} - \frac{1}{3}\int\frac{x+2}{x^2+x+1}\,dx$$ (05)

$$= \frac{1}{3}\int\frac{dx}{x-1} - \frac{1}{3}\int\frac{\frac{1}{2}(2x+1)+\frac{3}{2}}{x^2+x+1}\,dx$$ (05)

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{2}\int\frac{dx}{(x+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$ (05) (05) (05)

$$= \frac{1}{3}\ln|x-1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$ (05) (05)

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(d) $t = \tan(x/2) \Rightarrow dt = \frac{1}{2}(1+t^2)\,dx$

$\Rightarrow dx = \frac{2dt}{1+t^2}$ (05)

$$\int_0^{\pi/2} \frac{dx}{5+4\cos x+3\sin x} = \int_0^1 \frac{\frac{2}{(1+t^2)}\,dt}{5+4\left(\frac{1-t^2}{1+t^2}\right)+3\,\frac{2t}{1+t^2}}$$ (05)

$$= \int_0^1 \frac{2\,dt}{5(1+t^2)+4(1-t^2)+6t}$$

$$= \int_0^1 \frac{2dt}{t^2+6t+9}$$ (05)

$$= \int_0^1 \frac{2dt}{(t+3)^2} = 2\left[\frac{-1}{t+3}\right]_0^1 = 2\left[\frac{1}{3}-\frac{1}{4}\right] =$$ (05)

$$= \frac{1}{6}$$ 20

(16) If the two circles $x^2+y^2+2gx+2fy+c=0$ and $x^2+y^2+2g'x+2f'y+c'=0$ intersect orthogonally, then (05)

$(g-g')^2+(f-f')^2 = g^2+f^2-c+g'^2+f'^2-c'$ (05) (05)

$\Rightarrow 2gg'+2ff' = c+c'$ 15

Circle $C: x^2+y^2-8x-6y+16=0$

$(x-4)^2+(y-3)^2=3^2$

Circle of centre (4, 3) (05), Radius = 3 (05)

This circle touches the x - axis at (4, 0)

$\because$ y- coordinate of centre = 3 (05) 15

Circle $C_1: x^2+y^2=r^2$ touches C externally at A (r cosα, r sinα) where

$\cos\alpha = 4/5$, $\sin\alpha = 3/5$

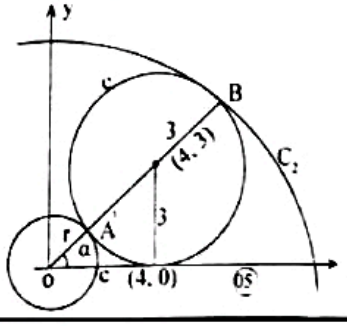
$r+3=5 \Rightarrow r=2$ (05)

$\because$ $A \equiv \left(\frac{8}{5}, \frac{6}{5}\right)$ (05)

Circle $C_2: x^2+y^2=R^2$ touches C internally at B. (05)

$R = 5+3 = 8$ (05)

$B \equiv (8\cos\alpha, 8\sin\alpha) = \left(\frac{32}{5}, \frac{24}{5}\right)$ (05)



05 30

Let S be a circle intersecting both C and C_1 or thogonally and touching the y - a xis

$s: x^2+y^2+2gx+2fy+c=0$

Its centre ; (-g, -f) and radius = $\sqrt{g^2+f^2-c}$

$\because$ S, touches the y - axis $\Rightarrow$ radius = $|g|$ (05)

$g^2+f^2-c = g^2 \Rightarrow f = \pm\sqrt{c}$ (05)

'S intersects c_1 orthogonally $\Rightarrow 0+0 = C-2^2$

$\Rightarrow C = 4$ and so $f = \pm 2$ (05)

(05)

Circle S intersects the given circle C or thogonally :

$\Rightarrow 2g(-4)+2f(-3) = 4+16 = 20$ (05)

$\Rightarrow 4g+3f+10=0$ (05)

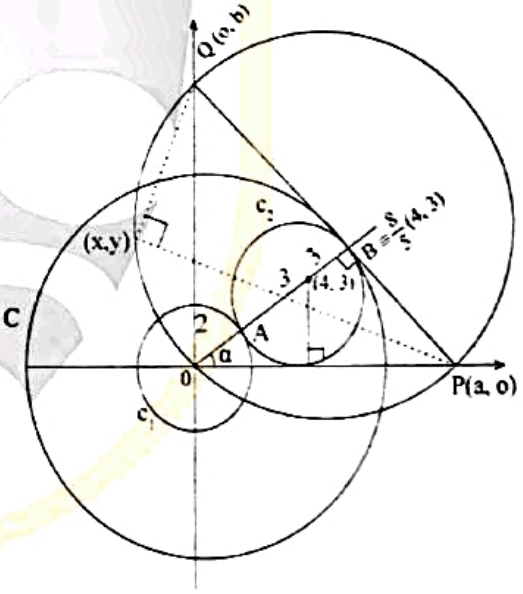
$f = +2 \Rightarrow 4g = -10-6 \Rightarrow g = -4$ (05)

$f = -2 \Rightarrow 4g = -10+6 \Rightarrow g = -1$ (05)

Two possible equations of S are

(05) $x^2+y^2-8x+4y+4=0$ and

$x^2+y^2-2x-4y+4=0$ (05) 50



Tangent common to C and C_2 has equation $\frac{x}{a}+\frac{y}{b}=1$, 10 where P ≡ (a, o) and Q ≡ (o, b) are the points where it meets the two axes of coordinates.

$a = 8\sec\alpha = 8\cdot\frac{5}{4} = 10 \Rightarrow P \equiv (10, 0)$ (05)

$b = 8\,\mathrm{cosec}\,\alpha = 8\cdot\frac{5}{3} = \frac{40}{3} \Rightarrow Q \equiv \left(0, \frac{40}{3}\right)$ (05)

Equation of PQ is $\frac{x}{10}+\frac{3y}{40} = 1 \Rightarrow 4x+3y=40$ (05)

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Aliter

Equation of the common tangent to the circles C and C_2 is given by

$(x^2 + y^2 - 8x - 6y + 16) - (x^2 + y^2 - 64) = 0$ (10)

$\Rightarrow 8x + 6y - 80 = 0 \Rightarrow 4x + 3y = 40$ (05)

There fore $P \equiv (10, 0)$ and $Q \equiv \left(0, \frac{40}{3}\right)$ (05) (05) 25

Any point (x, y) on the circle with line segment PQ as a diameter satisfies the condition

$\left(\frac{y-0}{x-a}\right)\left(\frac{y-b}{x-0}\right) = -1$ (05)

$x(x-a) + y(y-b) = 0$ (05)

i.e. $x^2 + y^2 - ax - by = 0$

$x^2 + y^2 - 10x - \frac{40}{3}y = 0$ (05)

$3(x^2 + y^2) - 30x - 40y = 0$ 40

(17)(a) $\cos^2(\alpha+\beta) + \cos^2\alpha + \cos^2\beta - 2\cos(\alpha+\beta)\cos\alpha\cos\beta = 1$

$= \cos(\alpha+\beta)[\cos(\alpha+\beta) - 2\cos\alpha\cos\beta] + \cos^2\alpha + \cos^2\beta$ (05) (05)

$= -[\cos\alpha\cos\beta - \sin\alpha\sin\beta][\cos\alpha\cos\beta + \sin\alpha\sin\beta]$ (05)

$+ \cos^2\alpha + \cos^2\beta$ (05)

$= -\cos^2\alpha\cos^2\beta + (1 - \cos^2\alpha)(1 - \cos^2\beta) + \cos^2\alpha + \cos^2\beta$ (05) (05)

$= 1$ 30

(b) $f(x) = \cos 2x + \sin 2x + 2(\cos x + \sin x) + 1$

$= 2\cos^2 x - 1 + 2\sin x\cos x + 2\cos x + 2\sin x + 1$ (05)

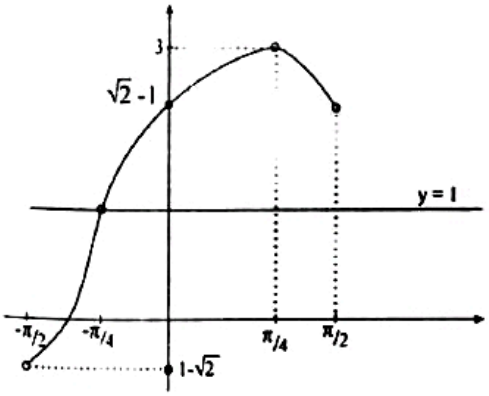
$= 2\cos x(\cos x + 1) + 2\sin x(\cos x + 1)$ (05)

$= 2\sqrt{2}(\cos x + 1)\sin(x + \pi/4)$ (05)

$k = 2\sqrt{2}$, $\alpha = \pi/4$ (05) (05) 25

$\frac{f(x)}{1+\cos x} = 2\sqrt{2}\sin(x + \pi/4) = \sqrt{2}\{g(x) - 1\}$ (05)

$y = g(x) = 2\sin(x + \pi/4) + 1$ (05)



Shape (05) Max (05) End points (05)

$x = 0, y = \sqrt{2} + 1$ (05) $y = 1$ (05)

$f(x) = 0 \Rightarrow g(x) = 1$ has only one solution (05)

$\therefore f(x) = 0$ has only one solution (05)

$\Rightarrow x = -\pi/4$ 45

(c) Sine Rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (05) when $A+B+C = \pi$ (05)

$\frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C}$ (05) (05)

$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$ (05)

$= \frac{\tan\frac{(B-C)}{2}}{\cot A/2}$ (05) — (1) ($\because A+B+C = \pi$)

$\frac{a}{b+c} = \frac{\sin A}{\sin B + \sin C} = \frac{2\sin A/2\ \cos A/2}{2\cos A/2\ \cos\frac{(B-C)}{2}}$ (05)

$= \frac{\sin A/2}{\cos\left(\frac{B-C}{2}\right)}$ — (2)

Multiplying the equation (1) and (2) we get

$\frac{a(b-c)}{(b+c)^2} = \frac{\tan\left(\frac{B-C}{2}\right)}{\cot A/2} \cdot \frac{\sin A/2}{\cos\left(\frac{B-C}{2}\right)}$ (05)

$\Rightarrow a(b-c) \cdot \frac{\cot A/2}{\sin A/2} = (b+c)^2 \frac{\tan\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}$ (05)

$\Rightarrow a(b-c)\operatorname{cosec} A/2 \cot A/2 = (b+c)^2 \tan\left(\frac{B-C}{2}\right)\sec\left(\frac{B-C}{2}\right)$ 50

Aliter (05)

(c) Sine Rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ when $A+B+C = \pi$ (05)

$\frac{a(b-c)}{(b+c)^2} = \frac{\sin A(\sin B - \sin C)}{(\sin B + \sin C)^2}$ (10) (05)

$= \frac{\sin A \cdot 2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{4\sin^2\left(\frac{B+C}{2}\right)\cos^2\left(\frac{B-C}{2}\right)}$ (05) (05)

$= \frac{\sin A\ \sin A/2 \sin\left(\frac{B-C}{2}\right)}{2\cos^2 A/2\ \cos^2\left(\frac{B-C}{2}\right)}$ (05) $\because A+B+C = \pi$

$= \frac{2\sin^2 A/2\ \cos A/2 \sin\frac{B-C}{2}}{2\cos^2 A/2\ \cos^2\left(\frac{B-C}{2}\right)}$

$= \frac{\sin A/2 \tan A/2 \tan\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}$ (05)

$= \sin A/2\ \tan A/2\ \tan\left(\frac{B-C}{2}\right)\sec\left(\frac{B-C}{2}\right)$ (05)

$\Rightarrow a(b-c)\operatorname{cosec} A/2\ \cot A/2 = (b+c)^2 \tan\left(\frac{B-C}{2}\right)\sec\left(\frac{B-C}{2}\right)$ 50

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(b) Length x of the road measured in metres

x	99.8	99.9	100.0	100.1	100.2	100.3	100.4
Frequency (f)	5	7	12	33	25	15	3

Transformation

$y = \frac{x - \bar{x}_0}{d} = \frac{x - 100.1}{0.1}$ (05)

extended table

y	-3	-2	-1	0	1	2	3	(10)
y^2	9	4	1	0	1	4	9	(05)

$\Sigma fy = -15 - 14 - 12 + 0 + 25 + 30 + 9 = -41 + 64 = 23$ (05)

Mean of y :

$\bar{y} = \frac{1}{100} \Sigma fy = 0.23$ (05)

$x = \bar{x}_0 + dy \Rightarrow \bar{x} = \bar{x}_0 + d\bar{y}$ (05)

$\therefore$ the mean of x is $\bar{x} = 100.1 + (0.1)\,0.23 = 100.123$ (05)

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Variance of y is

$S_y^2 = \frac{1}{100} \Sigma fy^2 - \bar{y}^2$ (05) and

$\Sigma fy^2 = 45 + 28 + 12 + 0 + 25 + 60 + 27 = 85 + 85 + 27 = 197$ (05)

$\frac{1}{100} \Sigma fy^2 = 1.97$, (05) $\bar{y}^2 = (0.23)^2 = 0.0529$ (05)

$\therefore$ Variance of y = 1.97 - 0.0529 = 1.917 (05)

Linear relation $x = dy + \bar{x}_0$

var $(x) = d^2$ var (Y), d = 0.1

$S_x^2 = d^2 S_y^2$ (05)

$\therefore$ var $(x) = (0.1)^2\, 1.917$

Taking square root of var (x)

Standard Deviation $= S_x = (0.1)\sqrt{1.917}$ (05)

$S_x = 0.1385$; $(\because \sqrt{1.917} \approx 1.385)$

Standard Deviation of x is 0.1385 m $\approx$ 0.139 m (05)

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