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G.C.E. (Advanced Level) Examination - August 2014
Combined Mathematics - I
Scheme of marking

Part A

(01) When n = 1, L.H.S = $\sum_{r=1}^{1} r(3r-1) = 2$ and,

R.H.S. = $1^2(1+1) = 2$

Hence, the result is true for n = 1 (05)

Take any $k \in \mathbb{Z}^+$ and assume that the result is true for n = k

i.e. $\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$ (05)

Now,

$$\sum_{r=1}^{k+1} r(3r-1) = \sum_{r=1}^{k} r(3r-1) + (k+1)\left[3(k+1)-1\right] \quad (05)$$

$$= k^2(k+1) + (k+1)(3k+2) \quad \text{(By the induction hypothesis)}$$

$$= (k+1)\left[k^2+3k+2\right]$$

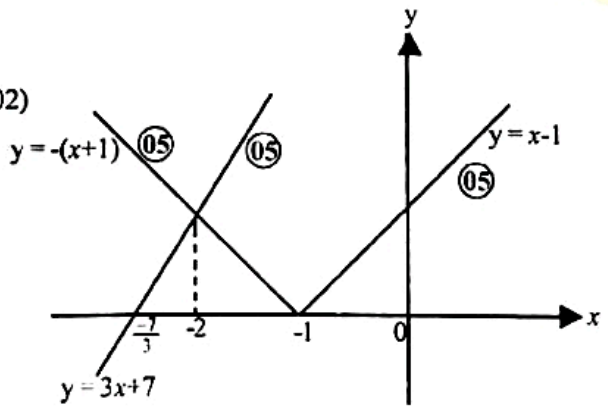
$$= (k+1)^2(k+2) \quad (05)$$

$$= (k+1)^2\left[(k+1)+1\right]$$

Hence, if the result is true for n = k, then it is also true for n = k+1, we have already Proved that the result is true for n = 1. Thus by the principle of mathematical induction, the result, is true for all $n \in \mathbb{Z}^+$ (05)

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(02)



At the point of inter section, we must have $-x - 1 = 3x + 7$ and so $x = -2$, (05)

Thus the solution set is $\{x \in \mathbb{R} : x < -2\}$ (05)

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Case (i) $\boxed{x \le -1}$ In this case $|x+1| > 3x+7$

$\Leftrightarrow -(x+1) > 3x+7$ (05) $\Leftrightarrow x < -2$ (05)

Thus, in this case x satisfying $x < -2$ are the solution

Case (ii) $x > -1$ In this case $|x+1| > 3x+7$

$\Leftrightarrow x+1 > 3x+7$ (05)

$\Leftrightarrow x < -3$ (05)

This contradiction implies that there are no solution in this case

Thus the solution set is $\{x \in \mathbb{R} : x < -2\}$ (05)

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Case (i) $\boxed{x \le -\frac{7}{3}}$

In this case $3x+7 \le 0$ *and hence* $|x+1| > 3x+7$ *is satisfied by all values of* x

satisfying $x \le -\frac{7}{3}$ (05)

case (ii) $\boxed{x > \frac{-7}{3}}$

In this case $|x+1| > 3x+7$

$\Leftrightarrow (x+1)^2 > (3x+7)^2$ (05)

$\Leftrightarrow 8x^2 + 40x + 48 < 0 \Leftrightarrow (x+2)(x+3) < 0 \Leftrightarrow -3 < x < -2$ (05)

In this case, the solutions are x values

Satisfying $\frac{-7}{3} < x < -2$ (05)

By combining the two cases the solution set is $\{x \in R : x < -2\}$ (05)

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The inequality $|x+1|>3x+7$ is equivalent to

$|x+1|-(3x+7)>0$

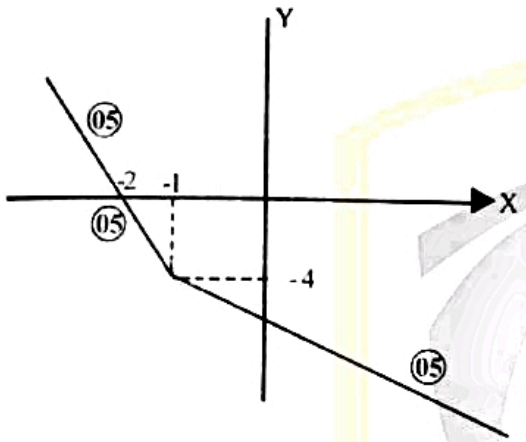
Let $y=|x+1|-(3x+7)$

Then,

$$y=\begin{cases}-2x-6, & \text{if } x\geq -1\\ -4x-8, & \text{if } x<-1\end{cases}$$ (05)

The solution set is

$\{x\in\mathbb{R}: x<-2\}$ (05)



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(03) Let z be the required complex number

Note that

$$A\hat{C}B=\frac{5\pi}{6}-\frac{\pi}{3}=\frac{\pi}{2}$$

Then, Since AB = 2

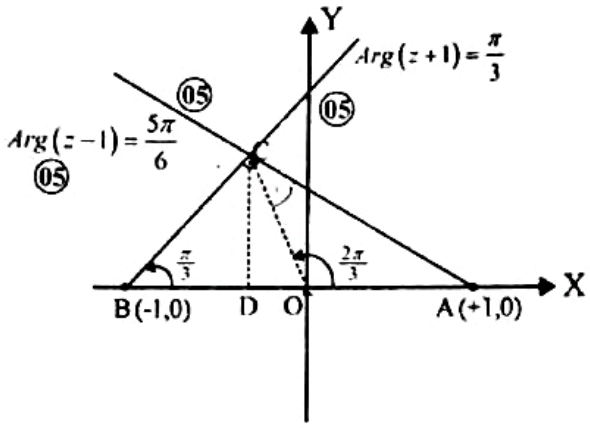
We have $BC=2\cos\frac{\pi}{3}=1$ (05)

Now $CD=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ and

$BD=\cos\frac{\pi}{3}=\frac{1}{2}$ (05)

$\left[or\ z+1=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]$

$\therefore z_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (05)



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Aliter

Note that

$$A\hat{C}B=\frac{5\pi}{6}-\frac{\pi}{3}=\frac{\pi}{2}$$ (05)

Thus C lies on the circle centered O with radius

$\therefore A\hat{O}C=\frac{2\pi}{3}$ (05) and hence $z_1=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

$=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (05)

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The equation of AC and BC are $y=\frac{-1}{\sqrt{3}}(x-1)$ (05) and

$y=\sqrt{3}(x+1)$ respectively.

By Solving them, We get $x=-\frac{1}{2}$ and $y=\frac{\sqrt{3}}{2}$ (05)

$\therefore z_1=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$ (05)

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(04) In the usual notation,

$\left(2+\frac{3}{x}\right)(1+x)^n=\left(2+\frac{3}{x}\right)\sum_{r=0}^{n}{}^nC_r\,x^r$ (05)

Since the coefficient of x^{n-2} is given to be 120.

We must have $2{}^nC_{n-2}+3{}^nC_{n-1}=120$ (05)

i.e. $2\frac{n!}{(n-2)!2!}+3\frac{n!}{(n-1)!\,1!}=120$ (05)

$\Leftrightarrow n(n-1)+3n=120$ (05)

$\Leftrightarrow n^2+2n-120=0\Leftrightarrow (n+12)(n-10)=0$

$n=10\left(\because n\in\mathbb{Z}^+\right)$ (05)

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(05)

$$\lim_{x\to 0}\frac{\tan^2 2x}{x\left(1-\sqrt{1+x}\right)}=\lim_{x\to 0}\frac{\tan^2 2x}{\left(1-\sqrt{1+x}\right)}\times\frac{\left(1+\sqrt{1+x}\right)}{\left(1+\sqrt{1+x}\right)}$$ (05)

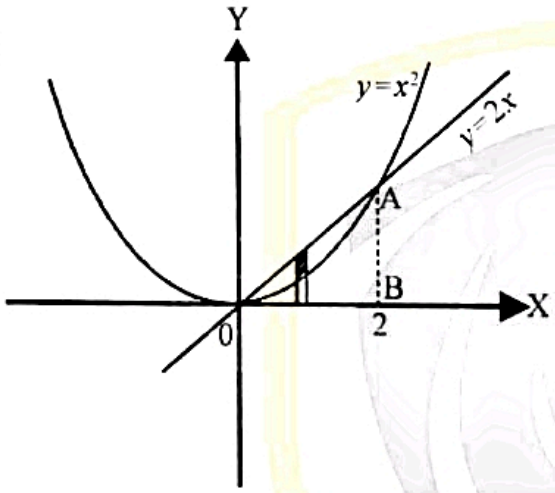
$$=\lim_{x\to 0}\frac{\sin^2 2x}{\cos^2 2x}\frac{\left(1+\sqrt{1+x}\right)}{\left(-x^2\right)}$$ (05)

$$=\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2\left(\frac{-4}{\cos^2 2x}\right)\left(1+\sqrt{1+x}\right)$$ (05)

$$=1^2\times\left(\frac{-4}{1}\right)\times 2=-8$$ (05) (05)

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(06)



At the points of intersections, we have $x^2 = 2x$ and so $x = 0$ or $x = 2$ (05)

The required area $= \int_0^2 (2x - x^2)\,dx$ (10)

$$=\left(x^2-\frac{x^3}{3}\right)\Big|_0^2=4-\frac{8}{3}$$ (05)

$= \frac{4}{3}$ squre units (05)

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Aliter

At the points of intersections, we have $x^2 = 2x$ and so $x = 0$ or $x = 2$. (05)

The required area $= Area\ \Delta\ OAB - \int_0^2 x^2 dx$ (10)

$$=\frac{1}{2}\times 2\times 4-\left|\frac{x^3}{3}\right|_0^2$$ (05)

$$=4-\frac{8}{3}=\frac{4}{3}$$ Squre Units (05)

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(07) $x = e^t + e^{-t} \rightarrow \frac{dx}{dt} = e^t - e^{-t}$

$y = e^t - e^{-t} \rightarrow \frac{dy}{dt} = e^t + e^{-t}$ } (05)

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{\left(e^t+e^{-t}\right)}{\left(e^t-e^{-t}\right)}$$ (05)

$$e^{\ln 2}=2,\quad e^{-\ln 2}=\frac{1}{2}$$

$$\therefore\ C\equiv\left[2+\frac{1}{2},\ 2-\frac{1}{2}\right]$$

Let C be the point Corresponding to $t = \ln 2$,

Then $C\equiv\left[2+\frac{1}{2},\ 2-\frac{1}{2}\right]=\left[\frac{5}{2},\ \frac{3}{2}\right]$ (05)

$$\therefore\ \left.\frac{dy}{dx}\right|_{\left[\frac{5}{2},\frac{3}{2}\right]}=\frac{5/2}{3/2}=\frac{5}{3}$$ (05)

The required equation is

$$y-\frac{3}{2}=\frac{5}{3}\left[x-\frac{5}{2}\right]$$ (05)

$5x - 3y - 8 = 0$

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(08) The equation of AB is

$(1+\lambda)x - 2(1-\lambda)y - 2(1-\lambda) = 0$

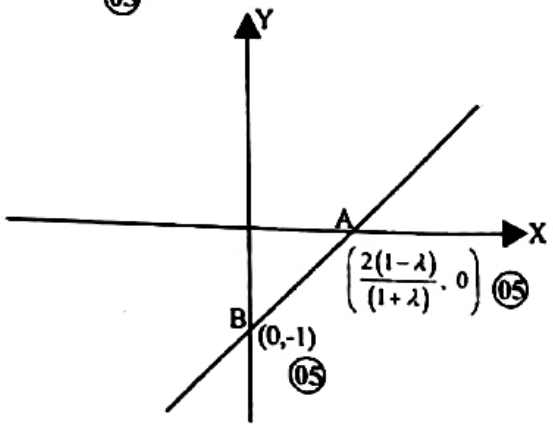
$x=\frac{2(1-\lambda)}{(1+\lambda)}$ when $y = 0$ and $y = -1$ when $x = 0$

Area Δ OAB $=\frac{1}{2}\times 1\times\left|\frac{2(1-\lambda)}{1+\lambda}\right|=4$ (05)

$$\leftrightarrow\left|\frac{(1-\lambda)}{1+\lambda}\right|=4$$

$$\leftrightarrow\left(\frac{(1-\lambda)}{(1+\lambda)}\right)=\pm 4\leftrightarrow 1-\lambda=4(1+\lambda)\ \text{ or }\ 1-\lambda=-4(1+\lambda)$$

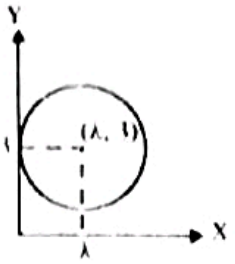
$\leftrightarrow \lambda=-\frac{3}{5}$ (05) or $\lambda=-\frac{5}{3}$ (05)



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(09)



The equation of the required circle can be

written as $(x-\lambda)^2+(y-3)^2=\lambda^2$ | Center (05)

i.e $x^2+y^2-2\lambda x-6y+9=0$ | Equation (05)

Since this is orthogonal to the circle

$x^2+y^2-8x+4y-5=0$, we have

$2(-4)(-\lambda)+2.2.(-3)=-5+9$ (05)

$\Leftrightarrow 8\lambda-12=4$

$\Leftrightarrow \lambda=2$ (05)

Hence, the equation of the required circle is

$x^2+y^2-4x-6y+9=0$ (05) **25**

(10) $\tan\alpha=-1$ and $\frac{3\pi}{2}<\alpha<2\pi \Rightarrow \alpha=\frac{7\pi}{4}$ (05)

$\Rightarrow \cos\alpha=\frac{1}{\sqrt{2}}$ and $\sin\alpha=-\frac{1}{\sqrt{2}}$

$\sin\beta=\frac{1}{\sqrt{5}}$ and $\frac{\pi}{2}<\beta<\pi \Rightarrow$

$\cos\beta=-\sqrt{1-\sin^2\beta}=-\frac{2}{\sqrt{5}}$ (10)

Now, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ (05)

$=\left(\frac{1}{\sqrt{2}}\right)\left(\frac{-2}{\sqrt{5}}\right)-\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{5}}\right)$

$=\frac{-2}{\sqrt{10}}+\frac{1}{\sqrt{10}}$

$=\frac{-1}{\sqrt{10}}$ (05) **25**

Part B

(11) (a) $f(x)=3x^3+5x^2+ax-1$

Since $(3x-1)$ is a factor of $f(x)$, by the Factor Theorem, $f\left(\frac{1}{3}\right)=0$ (10)

Now, $f\left(\frac{1}{3}\right)=3\times\frac{1}{27}+5\times\frac{1}{9}+a\times\frac{1}{3}-1$ and so (10)

$1+5+3a-9=0$

$a=1$ (05) **25**

$f(x)=3x^3+5x^2+x-1$

$=(3x-1)(x^2+2x+1)$ (10)

$=(3x-1)(x+1)^2$ (05)

This is of the required form with $k=1$ (05) **20**

$(3x-1)=3(x+1)-4$ (05)

This is of the required form with $b=3$ and $C=-4$

$\therefore f(x)=[3(x+1)-4](x+1)^2$

$=3(x+1)^3-4(x+1)^2$ (05)

$\therefore$ The required remainder $=-4(x+1)^2$ (05) **15**

(b) Suppose that zero is a root of $ax^2+bx+C=0$

Then, by substituting $x=0$, we get $C=0$ (05)

Since $ac\neq 0$, this is a contradiction

$\therefore$ zero is not a root of $ax^2+bx+C=0$ (05) **10**

Thus $\alpha\neq 0$ and $\beta\neq 0$

Also $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$ (10) (10)

Now, with $\lambda=\frac{\alpha}{\beta}$,

$ac=(\lambda+1)^2=ac\left(\frac{\alpha}{\beta}+1\right)^2=\frac{ac}{\beta^2}(\alpha+\beta)^2$ (05) (05)

$=\frac{ac}{\beta^2}\cdot\frac{b^2}{a^2}=b^2\frac{c}{a\beta^2}$ (05) (05)

$=b^2\frac{\alpha\beta}{\beta^2}=b^2\frac{\alpha}{\beta}=b^2\lambda$ (05) **45**

OR

$$\left[\frac{(\lambda+1)^2}{\lambda}=\frac{\left[\frac{\alpha}{\beta}+1\right]^2}{\frac{\alpha}{\beta}}=\frac{(\alpha+\beta)^2}{\alpha\beta}=\frac{\frac{b^2}{a^2}}{\frac{c}{a}}=\frac{b^2}{ac}\therefore ac(\lambda+1)^2=b^2\lambda\right]$$

(05) (05) (05) (05) (05)

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As before $\gamma \neq 0$ and $\delta \neq 0$ and $pr(\mu+1)^2 = \mu q^2$ (05)

$\frac{ac(\lambda+1)^2}{pr(\mu+1)^2} = \frac{b^2\lambda}{q^2\mu}$ and so $acq^2\mu(\lambda+1)^2 = prb^2\lambda(\mu+1)$

$acq^2 = prb^2 \Leftrightarrow \mu(\lambda+1)^2 = \lambda(\mu+1)^2$ (10)

$\Leftrightarrow \lambda^2\mu + 2\lambda\mu + \mu = \lambda\mu^2 + 2\lambda\mu + \lambda$

$\Leftrightarrow \lambda\mu(\lambda-\mu) - (\lambda-\mu) = 0$

$\Leftrightarrow (\lambda-\mu)(\lambda\mu-1) = 0$ (05)

$\Leftrightarrow \lambda = \mu$ or $\lambda = \frac{1}{\mu}$ (05)

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The roots are in the same ratio $\Leftrightarrow \lambda = \mu$ or $\lambda = \frac{1}{\mu}$ (05)

$\therefore$ We must have $acq^2 = prb^2$

i.e. $2k(6k)^2 = 8 \times 9$ and so $k^3 = 1$

$\therefore k = 1$ (05)

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(12) (*a*) (i) Thus the required number of ways

$= {}^6C_2 \times {}^6C_2 \times {}^6C_2 = (15)^3 = 3375$ (05)

(05) (10)

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(ii) The required number of ways

$= {}^6C_2 \times {}^4C_2 \times {}^2C_2 = 90$ (05)

(10)

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(iii) Number of different ways of selecting two players each from

two schools $= {}^6C_2 \times {}^3C_2 \times {}^3C_2$ (10)

The number of different ways of selecting one player each from two of the remaining schools

$= {}^4C_2 \times {}^3C_1 \times {}^3C_1$ (10)

Hence, the required number of ways

$= {}^6C_2 \times {}^3C_2 \times {}^3C_2 \times {}^4C_2 \times {}^3C_1 \times {}^3C_1 = 7290$ (05)

(05)

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(*b*) $U_r = \frac{r^2-r-5}{r(r+1)(r+4)(r+5)}$

$r^2 - r - 5 = A(r^2-1)(r+5) - Br^2(r+4)$

$= (A-B)r^3 + (5A-4B)r^2 - Ar - 5A$

(05)

Comparing the coefficients of

r^3,	0	=	A - B	→ (01)	(10)
r^2,	1	=	5A - 4B	→ (02)	
r,	-1	=	- A	→ (03)	
r^0,	-5	=	-5A	→ (04)	

(01) and (03) ⇨ A = 1, B = 1

Note that (02) and (04) are also satisfied by these values.

Hence, there exists constants A and B satisfying the given conditions. Two values are A = 1 and B = 1. (05)

(05)

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For $r \in \mathbb{Z}^+$

$u_r = \frac{r^2-r-5}{r(r+1)(r+4)(r+5)}$

$= \frac{(r^2-1)(r+5) - r^2(r+4)}{r(r+1)(r+4)(r+5)}$ (05)

So, $u_r = \frac{r-1}{r(r+4)} - \frac{r}{(r+1)(r+5)}$ (05)

$= f(r) - f(r+1)$ (05)

Where $f(r) = \frac{r-1}{r(r+4)}$ (05)

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When

r = 1	u_1	=	$f_{(1)}$ -	$f_{(2)}$ (05)
r = 2	u_2	=	$f_{(2)}$ -	$f_{(3)}$
...	...		...	
...	...		...	
r = n-1	u_{n-1}	=	$f_{(n-1)}$-	$f_{(n)}$ (05)
r = n	u_n	=	$f_{(n)}$ -	$f_{(n+1)}$

$\sum_{r=1}^{n} u_r = f_{(1)} - f_{(n+1)} = -\frac{n}{(n+1)(n+5)}$ (05)

(05)

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Since $\lim_{n\to\infty} \sum_{r=1}^{n} u_r = \lim_{n\to\infty} \frac{-n}{(n+1)(n+5)} = 0$

$\sum_{r=1}^{n} u_r$ is convergent (05)

Its sum is O (05)

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$\sum_{r=3}^{\infty} 3u_r = 3\left[\sum_{r=1}^{\infty} u_r - u_1 - u_2\right]$ (05)

$= 3[0 - f(1) + f(3)]$

$= 3 \times \frac{2}{3\times 7} = \frac{2}{7}$ (05)

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(13) (*a*) Since $A=\begin{pmatrix}1 & 0\\ 0 & a\\ 1 & 1\end{pmatrix}$, we have $A^T=\begin{pmatrix}1 & 0 & 1\\ 0 & a & 1\end{pmatrix}$ (05)

and hence, $A^TA=\begin{pmatrix}2 & 1\\ 1 & a^2+1\end{pmatrix}$ (10)

Now, $A^TA=B \Leftrightarrow \begin{pmatrix}2 & 1\\ 1 & a^2+1\end{pmatrix}=\begin{pmatrix}b & 1\\ 1 & 1\end{pmatrix}$

$\Leftrightarrow 2=b$ and $a^2+1=1$ (10)

$\Leftrightarrow b=2$ and $a=0$ (05) **30**

Since $C=\begin{pmatrix}7 & 5\\ 5 & 3\end{pmatrix}$ and $X=\begin{pmatrix}u\\ u+1\end{pmatrix}$, we have

$CX=\begin{pmatrix}12u+5\\ 8u+3\end{pmatrix}$ and $BX=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}u\\ u+1\end{pmatrix}=\begin{pmatrix}3u+1\\ 2u+1\end{pmatrix}$ (05) (05)

Hence, $CX=\lambda BX \Leftrightarrow 12u+5=\lambda(3u+1)$ and

$8u+3=\lambda(2u+1)$ (05)

$\frac{12u+5}{8u+3}=\frac{\lambda(3u+1)}{\lambda(2u+1)}$ and so

$24u^2+22u+5=24u^2+17u+3$ (05)

This gives us $u=-\frac{2}{5}$

Then $-\frac{16}{5}+3=\lambda\left(-\frac{4}{5}+1\right) \therefore \lambda=-1$ (05) **30**

Now, $C-\lambda B=\begin{pmatrix}7 & 5\\ 5 & 3\end{pmatrix}+\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}9 & 6\\ 6 & 4\end{pmatrix}$ (05)

Since $\begin{vmatrix}9 & 6\\ 6 & 4\end{vmatrix}=0$ (05) the inverse of C- λB does not exist. (05) **15**

Aliter

Now, $C-\lambda B=\begin{pmatrix}7 & 5\\ 5 & 3\end{pmatrix}+\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}9 & 6\\ 6 & 4\end{pmatrix}$ (05)

Suppose $\begin{pmatrix}9 & 6\\ 6 & 4\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (05)

$\Leftrightarrow 9p+6r=1 \rightarrow (01)$

$9q+6s=0 \rightarrow (02)$

$6p+4r=0 \rightarrow (03)$

$6q+4s=1 \rightarrow (04)$

$(01)\times\frac{2}{3} \Rightarrow 6p+4r=\frac{2}{3}$

$(03) \Rightarrow 6p+4r=0$

} This is a contradiction (05)

Therefore, the inverse of C- λB does not exist. **15**

(b) (i) $|1-z|^2=(1-z)\overline{(1-z)}$ (05)

$=(1-z)(1-\bar{z})$ (05)

$=1-(z+\bar{z})+z\bar{z}$

$=1-2\operatorname{Re}z+|z|^2$ (05) **15**

(i) For $|z|\neq 1$, $\frac{1}{1-z}=\frac{1}{1-z}\times\frac{\overline{(1-z)}}{(1-z)}$ (05) $=\frac{1-\bar{z}}{|1-z|^2}$ (05)

$\therefore \operatorname{Re}\frac{1}{1-z}=\frac{1-\operatorname{Re}\bar{z}}{|1-z|^2}=\frac{1-\operatorname{Re}z}{|1-z|^2}$ (05) (05) **20**

Aliter

Let $z=x+iy$ with $x, y\in\mathbb{R}$

(i) Then $1-z=1-x-iy$ (05)

$|1-z|^2=(1-x)^2+y^2=1-2x+x^2+y^2$ (05)

$=1-2\operatorname{Re}z+|z|^2$ (05) **15**

(ii) For $z\neq 1$, $\frac{1}{1-z}=\frac{1}{1-x-iy}\times\frac{(1-x)+iy}{(1-x)+iy}$ (05)

$=\frac{(1-x)+iy}{(1-x)^2+y^2}$ (05)

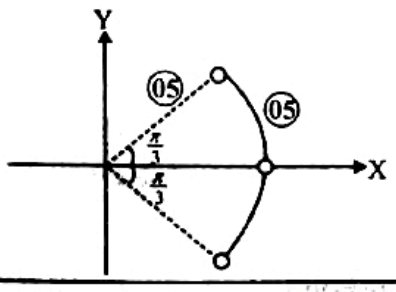
$\operatorname{Re}\frac{1}{1-z}=\frac{1-x}{(1-x)^2+y^2}=\frac{1-\operatorname{Re}z}{|1-z|^2}$ (05) (05) **20**

$\operatorname{Re}\frac{1}{1-z}=\frac{1}{2} \Leftrightarrow \frac{1-\operatorname{Re}z}{|1-z|^2}=\frac{1}{2}$ for $z\neq 1$

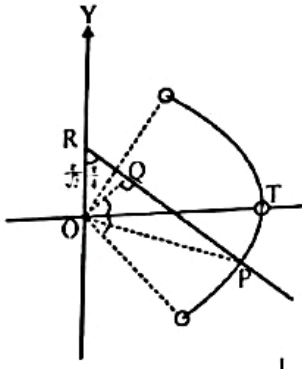
$\Leftrightarrow 2(1-\operatorname{Re}z)=1-2\operatorname{Re}z+|z|^2$ for $z\neq 1$ (05)

$\Leftrightarrow |z|^2=1$ *For* $Z\neq 1$

$\Leftrightarrow |z|=1$ *For* $Z\neq 1$ (05) **10**



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$\text{Re}\, z + \text{Im}\, z = \frac{1}{\sqrt{2}}$ (05)

Since OP = 1 and $OQ = \frac{1}{\sqrt{2}}$ $\sin\frac{\pi}{4} = \frac{1}{2}$, we have (05)

$Q\hat{O}P = \frac{\pi}{3}$

$\therefore T\hat{O}P = Q\hat{O}P - Q\hat{O}T = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$ (05)

$z = \cos\left(\frac{-\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)$

$= \cos\frac{\pi}{12} - i\sin\frac{\pi}{12}$ (05)

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$z \in s \Rightarrow z = \cos\theta + i\sin\theta; -\frac{\pi}{3} < \theta < \frac{\pi}{3}$ (05)

$\text{Re}\, z + \text{Im}\, z = \frac{1}{\sqrt{2}} \Rightarrow \cos\theta + \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = \frac{1}{2}$

$\Rightarrow \sin 2\theta = -\frac{1}{2}$ (05), since $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$; we have $\theta = -\frac{\pi}{12}$ (05)

$\therefore z = \cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right) = \cos\frac{\pi}{12} - i\sin\frac{\pi}{12}$ (05)

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(14) (a) Let $f(x) = \frac{8x}{(x+1)(x^2+3)}; x \neq -1$

So, for $x \neq -1$, we have

$f'(x) = \frac{(x+1)(x^2+3)\cdot 8 - 8x\left[(x^2+3)+(x+1)2x\right]}{(x+1)^2(x^2+3)^2}$ (15)

$= \frac{8(x^2+3) - 16x^2(x+1)}{(x+1)^2(x^2+3)^2} = \frac{8\left[3 - x^2 - 2x^2\right]}{(x+1)^2(x^2+3)^2}$

$= \frac{8(1-x)(2x^2+3x+3)}{(x+1)^2(x^2+3)^2}$ (10)

25

Turning Points $f'(x) = 0 \Leftrightarrow x = 1$, $(\because 2x^2+3x+3 \neq 0)$ (05) (05)

Note that $2x^2 + 3x + 3 > 0$ for all x and hence

$\frac{8(2x^2+3x+3)}{(x+1)^2(x^2+3)^2} > 0$ for all $x \neq 1$

Thus, the sign of $f'(x)$ for $x \neq \pm 1$ is the same as the sign of $(1 - x)$

	$-\infty < x < -1$	$-1 < x < 1$	$-1 < x < \infty$
Sing of $f'(x)$	(+)	(+)	(-)
	$f(x)$ is increasing	$f(x)$ is increasing	$f(x)$ is decreasing
	(05)	(05)	(05)

$f'(x)$ is underfined at $x = -1$ (05)

Hence, the graph of y = f(x) has only one turning point; It is a local maximum point with coordinates (1, 1) (05)

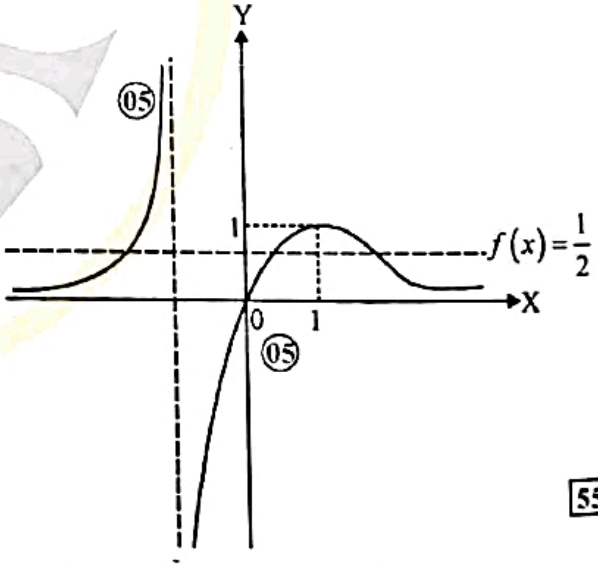
vertical asymptote: f(x) is underfined only at $x = -1$

Also $\lim_{x\to -1^-} f(x) = +\infty$ and $\lim_{x\to -1^+} f(x) = -\infty$

The only vertical asymptote is $x = -1$ (05)

Horfizontal asymptotes : $\lim_{x\to\pm\infty} f(x) = 0$

$\therefore$ y = 0 is the only horizontal asymptote. (05)



55

$(x+1)(x^3+3) = 16x \Leftrightarrow \frac{1}{2} = \frac{8x}{(x+1)(x^3+3)}$.

$\Leftrightarrow f(x) = \frac{1}{2}$ (05)

The required number of solutions is the number of points of intersection of the graph

$y = f(x)$ and $y = \frac{1}{2}$

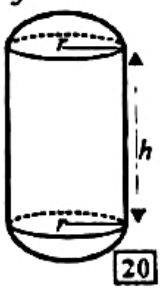
From the sketches of graphs, we see that this number is 3. (05)

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(b) Total volume of the composite body $= \frac{4}{3}\pi r^3 + \pi r^2 h$ (10)



It is given that

$$= \frac{4}{3}\pi r^3 + \pi r^2 h = 36\pi \quad (05)$$

$$\Rightarrow 4r^3 + 3r^2 h = 108 \quad (05)$$

$$\Rightarrow h = \frac{108 - 4r^3}{3r^2}$$

20

now $h > 0 \Rightarrow r < 3$, so we must have $0 < r < 3$ (05)

cost of material.

$$C = 300 + 2\pi rh + 1000 \times 4\pi r^2 \quad (05)$$

$$= 200\pi\left\{3r\left(\frac{108 - 4r^3}{3r^2}\right) + 20\,r^2\right\} (05)$$

$$= 800\pi\left[4r^2 + \frac{27}{r}\right] For\ 0 < r < 3 \qquad 15$$

$$\frac{dc}{dr} = 800\pi\left[8r - \frac{27}{r^2}\right] (05)$$

$$\frac{dc}{dr} = 0 \Leftrightarrow 8r = \frac{27}{r^2} \Leftrightarrow r = \frac{3}{2} \quad (05)$$

(05)

Note that $\frac{dc}{dr} < 0$ for $0 < r < \frac{3}{2}$ and

$\frac{dc}{dr} > 0$ for $\frac{3}{2} < r < 3$ (05)

Hence, C is minimum when $r = \frac{3}{2}$ (05) 25

(15) (a) $\int \frac{3x+2}{x^2+2x+5}\,dx$

$$= \int \frac{3(x+1)-1}{x^2+2x+5}dx \quad (05)$$

$$= \frac{3}{2}\int \frac{2x+2}{x^2+2x+5}dx - \int \frac{1}{(x+1)^2+4}dx \quad (05)$$

$$= \frac{3}{2}\ell n\left(x^2+2x+5\right) - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C, \text{ where}$$

(05) (05) (05)

C is an arbitrary constant.

[Note that $x^2 + 2x + 5 > 0$] 25

(b)

$$I = \int e^x \cos(\ell nx)\,dx$$

$$= \int e^x \cos(\ell nx)\frac{dx}{dx}\,dx \quad (05)$$

$$= \left[x\cos(\ell nx)\right]_1^e + \int x\sin(\ell nx)\frac{1}{x}dx$$

(05) (05) (05)

$$= e^e \cos(\ell ne^e) - \cos(\ell n\,1) + \int \sin \ell n(x)\frac{dx}{dx}dx \quad (05)$$

(05)

$$= e^e \cos\pi - \cos o - \left[x\sin(\ell nx)\right]\int_1^e - \int_1^e x\cos(\ell nx)\frac{1}{x}dx$$

(05) (05)

$$= -e^e - 1 - e^e Sin\,\pi - \sin(\ell n1) - I \quad (05)$$

$$2I = -e^e - 1$$

$$I = -\frac{1}{2}\left(e^e + 1\right) \quad (05)$$

50

(c) Let u = a-x, Then, x = a-u and $\frac{dx}{du} = -1 \Rightarrow dx = -du$

$u = a$ at $x = 0$ $u = a$ at $x = 0$ (05)

$$\int_0^a f(x)\,dx = \int_0^a f(a-u)(-du) \quad (05)$$

$$= \int_0^a f(a-u)\,du$$

$$= \int_0^a f(a-x)\,dx \quad (05)$$

15

$$P(x) = (x-\pi)(2x+\pi)$$

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{P(x)}\,dx = \int_0^{\pi/2} \frac{\sin^2\left(\pi/2 - x\right)}{P\left(\pi/2 - x\right)}\,dx \quad (05)$$ (by the above result)

$$\int_0^{\pi/2} \frac{\cos^2 x}{P\left(\pi/2 - x\right)}\,dx \quad (05) = \int_0^{\pi/2} \frac{\cos^2 x}{P(x)}\,dx$$

$$P\left(\pi/2 - x\right) = \left(\pi/2 - x - \pi\right)\left[2\left(\pi/2 - x\right) + \pi\right]$$

$$= -\frac{1}{2}(2x+\pi).2(\pi - x) = (x-\pi)(2x+\pi)$$

(10)

$$= P(x)$$

20

$$2I = \int_0^{\pi/2} \frac{Sin^2 x}{P(x)}\,dx + \int_0^{\pi/2} \frac{\cos^2 x}{P(x)}\,dx \quad (05)$$

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{Sin^2 x + \cos^2 x}{P(x)}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{P(x)}\,dx \quad (05)$$

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$$I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{(x-\pi)(2x+\pi)}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{\frac{1}{3\pi}}{(x-\pi)} - \frac{\frac{2}{3\pi}}{2x+\pi}\right] dx \quad (05)\ (05)$$

$$= \frac{1}{2}\left[\frac{1}{3\pi}\ell n|x-\pi| - \frac{2}{3\pi}\times\frac{1}{2}\ell n|2x+\pi|\right]_0^{\pi/2} \quad (05)$$

$$= \frac{1}{6\pi}\left\{\ell n\left|-\pi/2\right| - \ell n|-\pi| - \ell n|2\pi| + \ell n|\pi|\right\} \quad (05)$$

$$= \frac{1}{6\pi}\left(\ell n\ \pi/2 - \ell n 2\pi\right) \quad (05)$$

$$= \frac{1}{6\pi}\ell n\left|\frac{\pi/2}{2\pi}\right| = \frac{1}{6\pi}\ell n\left(\frac{1}{4}\right) \quad (05)$$

30

(16) Let m_1 and m_2 be the slopes of ℓ_1 and ℓ_2 respectively.

Then $m_1 = -2$ (05) and $m_2 = -\frac{1}{2}$ (05) Let θ be the acute angle between ℓ_1 and ℓ_2.

Then $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$ (05)

$$= \left|\frac{-2+\frac{1}{2}}{1+(-2)\left(-\frac{1}{2}\right)}\right| = \left|\frac{-\frac{3}{2}}{2}\right| = \frac{3}{4} \quad (05)$$

$\theta = \tan^{-1}\left(\frac{3}{4}\right)$

20

The angle bisectors are given by

$$\frac{|2x+y-5|}{\sqrt{5}} = \frac{|x+2y-4|}{\sqrt{5}} \quad (10)$$

i.e. $2x+y-5 = \pm(x+2y-4)$ (05)

$-x+y+1=0$ *or* $3x+3y-9=0$

$x-y-1=0$ *or* $x+y-3=0$ (05) (05)

Let the a cute angle between ℓ_1 and the line $x-y-1=0$ be α

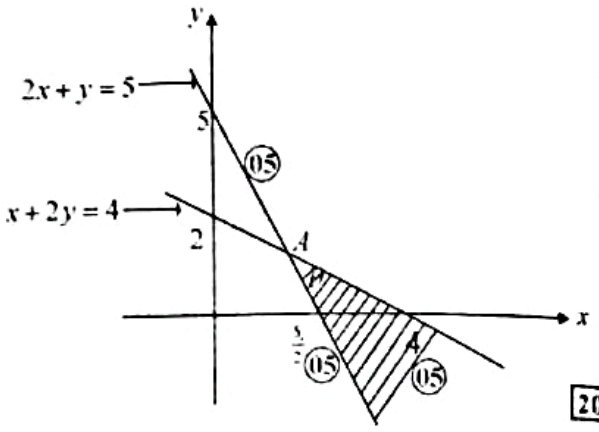
Then $\tan\alpha = \left|\frac{-2-1}{1+(-2)(1)}\right| = 3 > 1$ (05)

Thus, $x-y-1=0$ is not the required bisector.

$\therefore$ $x+y-3=0$ gives us the equation of the required bisector (05)

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Solving $2x+y=5$ and $x+2y=4$, simultaneously, we get $x=2$ and $y=1$, $A \equiv (2,1)$ (05)



20

The centre of S must lie on $x+y-3=0$

Let the coordinates of the centre of S be written as $(2+t, 1-t)$ (05)

Since the radius of S is $\sqrt{5}$, we have

$$\left|\frac{2(2+t)+(1-t)-5}{\sqrt{5}}\right| = \sqrt{5} \quad (10)$$

$|t| = 5$

$t = \pm 5$ (05)

$c = (7,-4)$ or $(-3,6)$ second point is not in R. (05) (05)

Equation of S :

$(x-7)^2+(y+4)^2 = 5$ (05)

$x^2-14x+49+y^2+8y+16=5$

$x^2+y^2-14x+8y+60=0$ (05)

40

Aliter

C = (t', 3-t') (05)

Since the radius of S is $\sqrt{5}$, we have

$$\frac{|2t'+(3-t'-5)|}{\sqrt{5}} = \sqrt{5} \quad (10)$$

$|t'-2| = 5$

$t' = 7$ *or* $t' = -3$ (05)

$C \equiv (7,-4)$ *or* $(-3,6)$: second point is not in R (05) (05)

$\therefore$ The equation of S.

$(x-7)^2+(y+4)^2 = 5$ (05)

$x^2-14x+49+y^2+8y+16=5$ (05)

$x^2+y^2-14x+8y+60=0$

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$x_0x + y_0y + g(x + x_0) + f(y + y_0) + c = 0$ (05)

With $x_0 = 2, y_0 = 1, g = -7, f = 4, c = 60$

gives us $2x + y - 7(x + 2) + 4(y + 1) + 60 = 0$

i.e. $-5x + 5y = -50$

$x - y = 10$ (05) **10**

The equation of the required circle can be written as

$x^2 + y^2 - 14x + 8y + 60 + \lambda(x - y - 10) = 0$ (10)

$A \equiv (2,1)$ is on this circle;

$\because 4 + 1 - 28 + 8 + 60 + \lambda(x - y - 10) = 0$ (05)

$45 - 9\lambda = 0$

$\lambda = 5$ (05)

There fore the required circle : $x^2 + y^2 - 9x + 3y + 10 = 0$ (05) **25**

(17) (a) $f(x) = \dfrac{1 - \tan x}{1 + \tan^2 x}; -\pi/2 < x < \pi/2$

$= \cos^2 x\left(1 - \dfrac{Sin\,x}{\cos x}\right) = \cos^2 x - Sin\,x\cos x$ (05)

$= \dfrac{1 + \cos 2x}{2}$ (05) $- \dfrac{Sin2x}{2}$ (05)

$= \dfrac{1}{2}(\cos 2x - Sin\,2x) + \dfrac{1}{2}$

$= \dfrac{1}{2}.\sqrt{2}\left[\dfrac{1}{\sqrt{2}}\cos 2x - \dfrac{1}{\sqrt{2}}Sin\,2x\right] + \dfrac{1}{2}$ (05)

$= \dfrac{1}{2}.\sqrt{2}\left[\cos \pi/4 \cos 2x - Sin\,\pi/4\, Sin2x\right] + \dfrac{1}{2}$ (05)

$= \dfrac{1}{\sqrt{2}}\left[\cos\left(2x + \pi/4\right)\right] + \dfrac{1}{2}$ (05)

$A = \dfrac{1}{\sqrt{2}},\ B = \dfrac{1}{2},\ \alpha = \pi/4$ (05) **35**

$f(x) = \dfrac{2 + \sqrt{2}}{4}$

$\dfrac{1}{\sqrt{2}}\cos\left(2x + \pi/4\right) + \dfrac{1}{2} = \dfrac{2 + \sqrt{2}}{4}$

$\cos\left(2x + \pi/4\right) = \dfrac{1}{2} = \cos \pi/3$ (05)

$2x + \pi/4 = 2n\pi \pm \pi/3 \quad n \in \mathbb{Z}$ (05)

$2x = 2n\pi \pm \pi/3 - \pi/4$

$2x = 2n\pi + \pi/12$ or $2x = 2n\pi - \pi/3 - \pi/4$

$2x = 2n\pi - 7\pi/12$

$x = n\pi + \pi/24$ or $x = n\pi - 7\pi/24$ (05)

$x = \pi/24,\ \dfrac{-7\pi}{24}\left(-\pi/2 < x < \pi/2\right)$ (05) **20**

$\dfrac{1 - \tan x}{1 + \tan^2 x} = \dfrac{2 + \sqrt{2}}{4}$

$4 - 4\tan x = 2 + \sqrt{2} + \left(2 + \sqrt{2}\right)\tan^2 x$

$\left(2 + \sqrt{2}\right) - \tan^2 x + 4\tan x - 2 + \sqrt{2} = 0 \rightarrow$

$\left(2 - \sqrt{2}\right)\tan^2 x + 4\tan x - \left(2 - \sqrt{2}\right) = 0 \rightarrow (01)$ (05)

$(01) \times \left(2 - \sqrt{2}\right)$

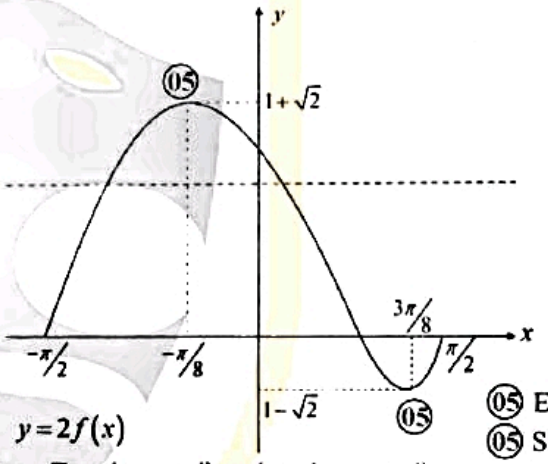
$2\tan^2 x + 4\left(2 - \sqrt{2}\right)\tan x - \left(2 - \left(\sqrt{2}\right)\right)^2 = 0$ (05)

$2\tan^2 x + 4k\tan x - k^2 = 0$, where $k = \left(2 - \sqrt{2}\right)$ (05) **15**

$\tan x = \dfrac{-4k \pm \sqrt{16k^2 + 8k^2}}{4} = \dfrac{-2k \pm \sqrt{6}k}{2}$ (05)

$\tan \pi/24 = -\left(2 - \sqrt{2}\right) + \dfrac{\sqrt{6}}{2}\left(2 - \sqrt{2}\right) \quad \because \left(\tan \pi/24 > 0\right)$ (05) (05)

$= \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$ (05) **20**



(05) End point
(05) Shape

$y = 2f(x)$

$= \sqrt{2}\cos\left(2x + \pi/4\right) + 1, \left(-\pi/2 < x < \pi/2\right)$ **20**

(b) Sin rule, $\dfrac{Sin\,A}{a} = \dfrac{Sin\,B}{b} = \dfrac{Sin\,C}{a}$ (05) **05**

$Sin\,2A + Sin\,2B + Sin\,2C = 2\,Sin\,(A+B)\cos\,(A-B) + 2\,Sin\,C\cos C$ (05)

$= 2\,Sin\,C.\cos\,(A-B) + 2\,Sin\,C\cos C$ (05)

$= 2\sin C\left[\cos(A-B) + \cos C\right]$ (05)

$= 2\,Sin\,C\left[\cos\,(A-B) - \cos\,(A+B)\right]$

$= 2\sin C.2\sin A\sin B$ (05)

$= 4\,Sin\,C\,.\,\dfrac{sin\,A}{a}\,\dfrac{sin\,B}{b}\,ab$ (05)

$Sin\,2A + Sin\,2B + Sin\,2C = 4\,Sin\,C\,\dfrac{Sin\,C}{c}\,.\,\dfrac{Sin\,C}{c}\,.ab$

$C^2(Sin\,2A + Sin\,2B + Sin\,2C) = 4\ Sin\,3C.\,ab$

$a : b : c = 1 : \lambda : \mu$ (05)

$\mu^2(sin2A + sin2B + sin2C) = 4\lambda sin^2 C.1.\lambda$

$\mu^2(sin2A + sin2B + sin2C) = 4\lambda sin^2 C$ (05) **35**