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G.C.E. (Advanced Level) Examination - August 2013
Combined Mathematics - I
Scheme of Marking

Part A

(01) When n = 1 L.H.S. $= \sum_{r=1}^{1}(2r+1) = 3$ and

R.H.S. $= 1\,(1+2) = 3$

Hence the result is true for n = 1 (05)

Take any $k \in \mathbb{Z}^+$ and assume that the result is true for n = k.

i.e. $\sum_{r=1}^{k}(2r+1) = k(k+2)$ (05)

Now $\sum_{r=1}^{k+1}(2r+1) = \sum_{r=1}^{k}(2r+1) + [2(k+1)+1]$ (05)

$= k(k+2) + 2k + 3$

$= k^2 + 4k + 3 = (k+1)(k+3)$ (05)

Hence if the result is true for n = k, it is also true for n = k + 1, We have already proved that the result is true for n = 1, By he Principle of Mathematical Induction the result is true for all $n \in \mathbb{Z}^+$. (05) **25**

(02) $\frac{2x+1}{3x-1} \geq 1 \Leftrightarrow \frac{2x+1}{3x-1} - 1 \geq 0 \Leftrightarrow \frac{(2x+1)-(3x-1)}{3x-1} \geq 0$

$\Leftrightarrow \frac{2-x}{3x-1} \geq 0$ (05)

Note $\frac{2-x}{3x-1}$ is O when $x = 2$ and undefined When $x = \frac{1}{3}$.

	$x < \frac{1}{3}$	$\frac{1}{3} < x < 2$	$2 < x$
$2 - x$	+	+	−
$3x-1$	−	+	+
$\frac{2-x}{3x-1}$	−	+	−
	(05)	(05)	(05)

The required solution set $= \left\{x \in \mathbb{R} : \frac{1}{3} < x \leq 2\right\}$ (05).

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(03) Let a_n be the nth tern of the series and let

$S_n = 6 - \frac{2^{n+1}}{3^{n-1}}$ for $n \in \mathbb{Z}^+$

Then $a_n = S_n - S_{n-1}$ (05)

$= \left[6 - \frac{2^{n+1}}{3^{n-1}}\right] - \left[6 - \frac{2^n}{3^{n-2}}\right]$ (05)

$= \frac{2^n}{3^{n-2}}\left[-\frac{2}{3} + 1\right]$

$= \frac{2^n}{3^{n-1}} = 2 \times \left(\frac{2}{3}\right)^{n-1}$ (05)

$\therefore \sum_{n=1}^{\infty} a_n$ is a geometric series with 2 as its first term and $\frac{2}{3}$ as its common ratio (05) since $\left|\frac{2}{3}\right| < 1$, the series is convergent. (05) **25**

(04) By the binomial theorem,

$\left(x + \frac{a}{x^3}\right)^{20} = \sum_{r=0}^{20} {}^{20}C_r\, x^{20-r}\left(\frac{a}{x^3}\right)^r$ (05) where

${}^{20}C_r = \frac{20!}{r!(20-r)!}$ for $r = 0, 1, \ldots 20$

$= \sum_{r=0}^{20} {}^{20}C_r\, a^r\, x^{20-4r}$

The term Independent of x is given by 20 - 4r = 0, and so r = 5, Since it is given to be $\frac{969}{2}$ (05), we have

${}^{20}C_5\, a^5 = \frac{969}{2}$ (05)

$\Leftrightarrow \frac{20!}{15!\,5!} a^5 = \frac{969}{2} \Leftrightarrow a^5 = \frac{1}{32} \Leftrightarrow a = -\frac{1}{2}$ (05) (05)

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(05) $\lim_{x\to 0}\dfrac{1-\cos x}{\sqrt{1+x^2}-\sqrt{1-x^2}}$

$= \lim_{x\to 0}\dfrac{1-\cos x}{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)}\times\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$ (05)

$= \lim_{x\to 0}\dfrac{2\sin^2\frac{x}{2}}{2x^2}\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)$ (05)

$= \lim_{x\to 0}\dfrac{\sin^2\frac{x}{2}}{4\left(\frac{x}{2}\right)^2}\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)$ (05)

$= \dfrac{1}{4}\times(1)^2\times(2)$ (05)

$= \dfrac{1}{2}$ (05)

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(06) $\dfrac{d}{dx}\left\{x\ell n\left(x+\sqrt{x^2+1}\right)\right\}$

$= \dfrac{x}{\left(x+\sqrt{x^2+1}\right)}\left[1+\dfrac{1}{2}\cdot\dfrac{2x}{\sqrt{x^2+1}}\right]+\ell n\left[x+\sqrt{x^2+1}\right]$ (05) (05)

$= \dfrac{x}{x+\sqrt{x^2+1}}\cdot\dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}+\ell n\left(x+\sqrt{x^2+1}\right)$

$= \dfrac{x}{\sqrt{x^2+1}}+\ell n\left(x+\sqrt{x^2+1}\right)$ (05)

Since $\dfrac{d}{dx}\sqrt{x^2+1}=\dfrac{x}{\sqrt{x^2+1}}$

we have

$\dfrac{d}{dx}\left\{x\ell n\left(x+\sqrt{x^2+1}\right)-\sqrt{x^2+1}\right\}=\ell n\left(x+\sqrt{x^2+1}\right)$ (05)

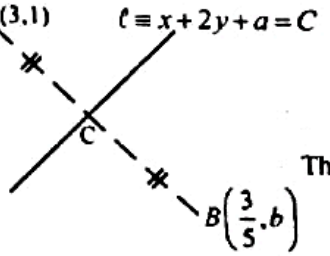
Therefore

$\therefore \int \ell n\left(x+\sqrt{x^2+1}\right)dx = x\ell n\left(x+\sqrt{x^2+1}\right)-\sqrt{x^2+1}+C,$ (05)

Where C is an arbitoary constant.

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(07)



Let C be the mid point of AB.

Then $C \equiv \left[\dfrac{3+\frac{3}{5}}{2}, \dfrac{1+b}{2}\right]$ (05)

Since C is on ℓ, we have

$\dfrac{9}{5}+(1+b)+a=0$ (05)

$\therefore a+b=-\dfrac{14}{5} \rightarrow (01)$

Since $AB\perp\ell$ we have $\left(\dfrac{b-1}{\frac{3}{5}-3}\right)\times\left(-\dfrac{1}{2}\right)=-1$

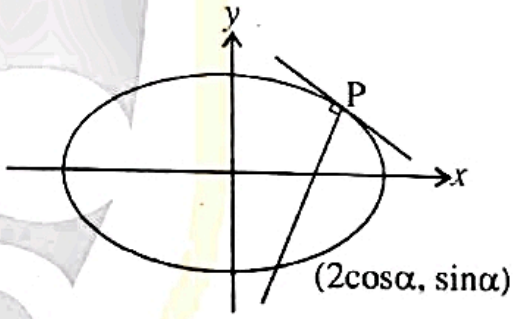
$\Rightarrow b-1=-\dfrac{24}{5}$

$\Rightarrow \quad b=-\dfrac{19}{5}$ (05)

Now by (01) $a=1$ (05)

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(08)



$P \equiv \left(\sqrt{2}, \dfrac{1}{\sqrt{2}}\right)$

$\dfrac{dx}{d\theta}=-2\sin\theta, \quad \dfrac{dy}{d\theta}=\cos\theta \Rightarrow \dfrac{dy}{dx}=\dfrac{-\cos\theta}{2\sin\theta}$ (05) (05)

Slope of the tangent at P is $=-\dfrac{1}{2}\cot\dfrac{\pi}{4}=-\dfrac{1}{2}$

$\therefore$ The slope of the normal line at P is 2 (05)

The equation of the normal line at P is $y-\dfrac{1}{\sqrt{2}}=2\left(x-\sqrt{2}\right)$ (05)

Since $(2\cos\alpha, \sin\alpha)$ is on this line,

$\sin\alpha-\dfrac{1}{\sqrt{2}}=2\left(2\cos\alpha-\sqrt{2}\right)$

$2\sin\alpha-8\cos\alpha+4\sqrt{2}-\sqrt{2}=0$

$2\sin\alpha-8\cos\alpha+3\sqrt{2}=0$ (05)

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(09) Since the required centre lies on $x + y = 0$, its coordinates are $(t, -t)$ for some $t \in \mathbb{Z}^+$

The equation of the circle with centre $(t, -t)$ and radius is

$(x - t)^2 + (y + t)^2 = 1$ (05)

or $x^2 + y^2 - 2tx + 2ty + 2t^2 - 1 = 0$ (05)

Since this is orthogonal to $x^2 + y^2 + 4y + 3 = 0$ (05)

We have $2(t)(2) = (2t^2 - 1) + 3$

i.e. $2t^2 - 4t + 2 = 0$

$(t - 1)^2 = 0$, $t = 1$, Required centre is $(1, -1)$ (05) (05) **25**

(10) Since $\cos^2\theta + \sin^2\theta = 1$ We have $\cos^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$ (05)

and so $\cos\theta = \pm\frac{2\sqrt{2}}{3}$

Since $\pi < \theta < \frac{3\pi}{2}, \cos\theta < 0$

$\cos\theta = \frac{-2\sqrt{2}}{3}$ (05)

Now $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{-2\sqrt{2}}{3}\right)\left(-\frac{1}{3}\right) = \frac{4\sqrt{2}}{9}$ (05)

Also $\cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2}{9} = \frac{7}{9}$ (05)

Thus, $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \left(\frac{4\sqrt{2}}{9}\right)\left(\frac{9}{7}\right) = \frac{4\sqrt{2}}{7}$ (05)

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Part B

(11) (a) Since $x - 1$ is a factor of $f(x)$: $f(1) = a + b - 11 + 6 = 0$ (10)

$\Rightarrow a + b = 5 \rightarrow (01)$

Since -6 is the remainder when $f(x)$ is divided by $x - 4$,

$f(4) = 64a + 16b - 44 + 6 = -6 \Rightarrow 4a + b = 2 \rightarrow (02)$ (10)

(01) and (02) $\Rightarrow$ $a = -1$ and $b = 6$, (05) **25**

Now $f(x) = -x^3 + 6x^2 - 11x + 6$

$= (x - 1)(-x^2 + 5x - 6)$ (10)

$= -(x - 1)(x^2 - 5x + 6)$

$= -(x - 1)(x - 2)(x - 3)$ (05)

Therefore the other linear factors are $x - 2$ and $x - 3$ (05) **20**

(b) (i) Since α and β are roots of $x^2 + bx + c = 0$

$\alpha + \beta = -b$ and $\alpha\beta = c$ (05)

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = b^2 - 4c$ (05) (05) **20**

By comparison $(\gamma - \delta)^2 = m^2 - 4n$ (05) **05**

Since $\alpha + \gamma = \beta + \delta \Rightarrow (\alpha - \beta)^2 = (\gamma - \delta)^2$ (05)

$\Rightarrow b^2 - 4c = m^2 - 4n$ (05) (05)

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(ii) $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$ (05)

$= [\alpha^2 + m\alpha + n][\beta^2 + m\beta + n]$ (05)

$= \alpha^2\beta^2 + \alpha\beta(\alpha + \beta)\, m + n(\alpha^2 + \beta^2) + \alpha\beta m^2 + mn(\alpha + \beta) + n^2$ (05)

$= c^2 - bcm + n(b^2 - 2c) + cm^2 - mnb + n^2$ (05)

$= (c^2 - 2cn + n^2) + (b^2n - bcm - mnb + cm^2)$

$= (c - n^2) + (b - m)(bn - cm)$ (05) **25**

$x^2 + bx + c = 0$ and $x^2 + mx + n = 0$ have a common root if and only if

$\alpha = \gamma$ or δ or $\beta = \gamma$ or δ (05)

$\leftrightarrow (\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$ (05)

$\leftrightarrow (c - n)^2 + (b - m)(bn - cm) = 0$ (05)

$\leftrightarrow (c - n)^2 = (m - b)(bn - cm)$ (05) **20**

$x^2 + 10x + k = 0$ and $x^2 + kx + 10 = 0$ have a common roof if and only if $(c - n)^2 = (m - b)(bn - cm)$ (05)

since $b = 10$, $c = k$, $m = k$ and $n = 10$,

$(k - 10)^2 = (k - 10)(100 - k^2)$ By simplifying, (05)

$0 = (k - 10)(k - 10 - 100 + k^2)$

$= (k - 10)(k^2 + k - 110)$

$= (k - 10)(k - 10)(k + 11)$ (05)

Hence $k = 10$ or $k = -11$ (05)

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(12) (a)

S	A	C
3	5	7

(i) $^{15}C_6 = \frac{15\times14\times13\times12\times11\times10}{6\times5\times4\times3\times2\times1} = 5005$ (05) (05) (05) (05)

15

(ii) Number of different ways that contain the two particular students $= {}^{13}C_4 = 715$ (10) (05)

Number of different ways that do not contain the paricular students together.

$= {}^{15}C_6 - {}^{13}C_4$ (05)
$= 5005 - 715$
$= 4290$ (05)

25

(iii) Number of different ways with two from each stream. $= {}^3C_2 \times {}^5C_2 \times {}^7C_2$ (10)
$= 3\times10\times21$
$= 630$ (05)

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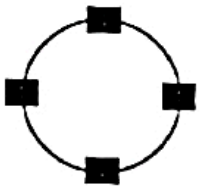
(iv) Number of different ways in which the 2 arts and the 2 commerce (05)

Students (i.e. 4 non-science student) can be seated in circular table = 3!

There are 4 gaps to which the 2 Science students can be accommodated.

Number of different ways for this is $= {}^4C_2 \times 2$ (05)

∴ Required number of different ways



$= 3! \times {}^4C_2 \times 2$ (05)
$= 72$ (05)

20

Aliter :-

Number of different ways in which all 6 can be seated in a circular table $= 5!$ (05)

Number of different ways in which all 6 with two from Science together $= 4! \times 2$ (05)

∴ Required number of different ways $= 5! - 4! \times 2$ (05)
$= 120 - 48$ (05)
$= 72$

20

(b) Let $U_r = \frac{3(6r+1)}{(3r-1)^2(3r+2)^2}$

$\frac{3(6r+1)}{(3r-1)^2(3r+2)^2} = \frac{A}{(3r-1)^2} + \frac{B}{(3r+2)^2}$ (05) for r

$\leftrightarrow$ $3(6r+1) = (A+B)9r^2 + (12A-6B)r + (4A+B)$ for $r \in \mathbb{Z}^+$

Equating coefficients of r^2 : $A + B = 0$
Equating coefficients of r^1 : $12A - 6B = 18$ (10)
Equating coefficients of r^0 : $4A + B = 3$
∴ $A = 1$ and $B = -1$ (05) (05)

25

Let $f(r) = \frac{1}{(3r-1)^2}$ then $f(r+1) = \frac{1}{(3r+2)^2}$

and $U_r = f(r) - f(r+1)$

When

$r = 1$ $u_1 = f(1) - f(2)$
$r = 2$ $u_2 = f(2) - f(3)$ (05)
.....
.....
$r = n-1$ $u_{n-1} = f(n-1) - f(n)$
$r = n$ $u_n = f(n) - f(n+1)$ (05)

$u_1 + u_2 + \ldots\ldots + u_n = f(1) - f(n+1)$ (05)

and $\therefore S_n = \frac{1}{4} - \frac{1}{(3n+2)^2}$ (05)

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yes (05)

05

$\lim_{n\to\infty} S_n = \frac{1}{4}$ (05) since $\lim_{n\to\infty} \frac{1}{(3n+2)^2} = 0$ (05)

∴ Therefore, infinite series is convergent

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(12) (a)

S	A	C
3	5	7

(i) ${}^{15}C_6 = \frac{15\times14\times13\times12\times11\times10}{6\times5\times4\times3\times2\times1} = 5005$ (05) (05) (05) (05)

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Number of different ways that do not contain the paricular students together.

$= {}^{15}C_6 - {}^{13}C_4$ (05)
$= 5005 - 715$
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(iii) Number of different ways with two from each stream.

$= {}^3C_2 \times {}^5C_2 \times {}^7C_2$ (10)
$= 3\times10\times21$
$= 630$ (05)

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(iv) Number of different ways in which the 2 arts and the 2 commerce (05)

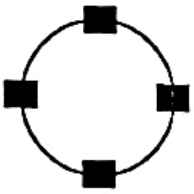
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There are 4 gaps to which the 2 Science students can be accommodated.

Number of different ways for this is $= {}^4C_2 \times 2$ (05)

∴ Required number of different ways

$= 3! \times {}^4C_2 \times 2$ (05)
$= 72$ (05)



20

Aliter :-

Number of different ways in which all 6 can be seated in a circular table $= 5!$ (05)

Number of different ways in which all 6 with two from Science together $= 4! \times 2$ (05)

∴ Required number of different ways $= 5! - 4! \times 2$ (05)
$= 120 - 48$ (05)
$= 72$

20

(b) Let $U_r = \frac{3(6r+1)}{(3r-1)^2(3r+2)^2}$

$\frac{3(6r+1)}{(3r-1)^2(3r+2)^2} = \frac{A}{(3r-1)^2} + \frac{B}{(3r+2)^2}$ (05) for $r \in \mathbb{Z}^+$

$\leftrightarrow$ $3(6r+1) = (A+B)9r^2 + (12A-6B)r + (4A+B)$ for $r \in \mathbb{Z}^+$

Equating coefficients of r^2 : $A + B = 0$
Equating coefficients of r^1 : $12A - 6B = 18$ (10)
Equating coefficients of r^0 : $4A + B = 3$
∴ $A = 1$ and $B = -1$ (05) (05)

25

Let $f(r) = \frac{1}{(3r-1)^2}$ then $f(r+1) = \frac{1}{(3r+2)^2}$

and $Ur = f(r) - f(r+1)$

When

$r = 1$ $u_1 = f(1) - f(2)$
$r = 2$ $u_2 = f(2) - f(3)$ (05)
.....
.....
$r = n-1$ $u_{n-1} = f(n-1) - f(n)$
$r = n$ $u_n = f(n) - f(n+1)$ (05)

$u_1 + u_2 + \ldots\ldots + u_n = f(1) - f(n+1)$ (05)

and ∴ $S_n = \frac{1}{4} - \frac{1}{(3n+2)^2}$ (05)

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yes (05)

05

$\lim_{n\to\infty} S_n = \frac{1}{4}$ (05) since $\lim_{n\to\infty} \frac{1}{(3n+2)^2} = 0$ (05)

∴ Therefore, infinite series is convergent

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Now $\left|S_n - \frac{1}{4}\right| = \frac{1}{(3n+2)^2} < 10^{-6}$ (05)

$\Leftrightarrow$ $(3n+2)^2 > 10^6$ and $n \in \mathbb{Z}^+$

$\Leftrightarrow$ $(3n+2) > 10^3$ and $n \in \mathbb{Z}^+$

$\Leftrightarrow$ $3n > 998$ and $n \in \mathbb{Z}^+$ (05)

$\Leftrightarrow$ $n > 332\frac{2}{3}$ and $n \in \mathbb{Z}^+$

$\therefore$ Therefore, the lower value of n = 333 (05) **15**

(13) (a) $Q^TQ = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I$ (05)

$\therefore Q^TQ = \lambda I$ Where $\lambda = 2$ (05) **10**

$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2I \Rightarrow$

$\Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = I$ (10)

$\therefore P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (05) **15**

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (05)

$AP = PD \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$ (05)

$\Leftrightarrow$ $\begin{aligned} \frac{a+b}{\sqrt{2}} &= \frac{2}{\sqrt{2}} \\ \frac{-a+b}{\sqrt{2}} &= \frac{-8}{\sqrt{2}} \\ \frac{c+d}{\sqrt{2}} &= \frac{2}{\sqrt{2}} \\ \frac{-c+d}{\sqrt{2}} &= \frac{8}{\sqrt{2}} \end{aligned}$ (05) $\Leftrightarrow$ $\begin{aligned} a+b &= 2 \\ -a+b &= -8 \\ c+d &= 2 \\ -c+d &= 8 \end{aligned}$

$\Rightarrow$ $a = 5, b = -3, c = -3, d = 5$ (10)

$\therefore A = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$ (05) **30**

Aliter :

$AP = PD \Rightarrow A = PDP^{-1}$ (10)

$\therefore A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (10)

$= \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$ (10) **30**

(b) $|z| = \sqrt{x^2+y^2}$ (05) and $\bar{z} = x - iy$ (05) **10**

$|z|^2 = x^2 + y^2 = (x+iy)(x-iy) = z\bar{z}$ (05)

$z - \bar{z} = (x+iy) - (x-iy) = 2iy = 2i\,\text{Im}\,z$ (05) **20**

$|z-3i|^2 = (z-3i)\overline{(z-3i)}$ (05)

$= (z-3i)(\bar{z}+3i)$ (05)

$= z\bar{z} + 3i(z-\bar{z}) + 9$ (05)

$= |z|^2 - 6\,\text{Im}\,z + 9$ (05)

$|1+3iz|^2 = (1+3iz)\overline{(1+3iz)}$

$= (1+3iz)(1-3i\bar{z})$

$= 1 + 3i(z-\bar{z}) + 9z\bar{z}$

$= 9|z|^2 - 6\,\text{Im}\,z + 1$ (10) **30**

Now

$|z - 3i| > |1 + 3iz|$ (05)

$\Leftrightarrow$ $|z - 3i|^2 > |1 + 3i|^2$

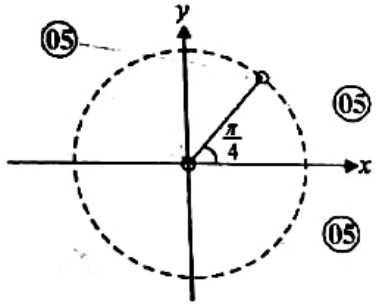
$\Leftrightarrow$ $|z|^2 - 6\text{Imz} + 9 > 9|z|^2 - 6\,\text{IMz} + 1$ (05)

$\Leftrightarrow$ $8(|z|^2 - 1) < 0$

$\Leftrightarrow$ $|z|^2 < 1$ (05)

$\Leftrightarrow$ $|z| < 1$ (05) **20**

$|z-3i| > |1+3iz| \Leftrightarrow |z| < 1$



(05) (05) (05) **15**

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(14) (a) Let $f(x)=\frac{x^2}{x^3-1}$; $x \neq 1$

$$f'(x)=\frac{\left(x^3-1\right)2x-x^2 3x^2}{\left(x^3-1\right)^2} \quad (10)$$

$$=\frac{-x^4-2x}{\left(x^3-1\right)^2}=\frac{-x\left(x^3+2\right)}{\left(x^3-1\right)^2} \quad (05) \qquad [15]$$

$f(x) = 0$ (05) $\leftrightarrow$ $x = 0$ or $x^3 = -2$

$\Rightarrow$ $x = 0$ or $x=(-2)^{\frac{1}{3}}$ (05)

x	$x<(-2)^{\frac{1}{3}}$	$(-2)^{\frac{1}{3}}<x<0$	$0<x<1$	$x>1$
Sign of $f'(x)$	(-)	(+)	(-)	(-)
	$f(x)$ is decreasing	$f(x)$ is increasing	$f(x)$ is decreasing	$f(x)$ is decreasing

(15)

at $x = 0$; $f(c) = 0$

at $x=(-2)^{\frac{1}{3}}$; $f(-2)^{\frac{1}{3}}=\frac{\left(-2^{\frac{1}{3}}\right)^2}{-2-1}=-\frac{4^{\frac{1}{3}}}{3}$

$\therefore$ The turning Points are $(0, 0)$ and $\left(-2^{\frac{1}{3}}, \frac{-4^{\frac{1}{3}}}{3}\right)$ (05) (05)

[35]

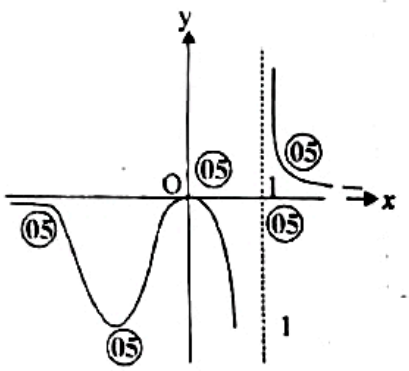
$f(x) \to 0$ as $x \to \infty$

and $f(x) \to 0$ as $x \to -\infty$

$f(x)$ is not defined at $x = 1$

$f(x) \to -\infty$ as $x \to 1^-$

and $f(x) \to \infty$ as $x \to 1^+$



[25]

(b) Perimeter : $P = 18x + 4y \rightarrow (01)$ (05)

Area $800 = 4xy + 8x^2 \rightarrow (02)$ (05)

By (2) $y=\frac{200}{x}-2x$ (05)

$\therefore P = 18x + 4\left(\frac{200}{x}-2x\right)$

$=10x+\frac{800}{x}$ (05) [20]

$y=\frac{2\left(100-x^2\right)}{x}$

Since $y > 0$ and $x > 0$ (05)

$100 - x^2 > 0 \Rightarrow x^2 < 100 \Rightarrow x < 10 \ (\because x > 0)$

$\therefore 0 < x < 10$ (05) [15]

(05)

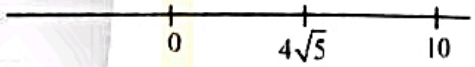
$\frac{dp}{dx}=10-\frac{800}{x^2}$ (10)

(05) $\frac{dp}{dx}=0 \Leftrightarrow 10-\frac{800}{x^2}=0$ (05)

$\Leftrightarrow x^2 = 80$

$\Leftrightarrow x = 4\sqrt{5}$ $\quad 0 < x < 10$ [25]

(05)



x	$0<x<4\sqrt{5}$	$4\sqrt{5}<x<10$
sign of $\frac{dp}{dx}=10-\frac{800}{x^2}$	(-)	(+)
	P is decreasing	P is increasing

The minimum Value of P occures at $x = 4\sqrt{5}$

Hence, the minimum value of the Perimeter is (10)

$=10\times 4\sqrt{5}+\frac{800}{4\sqrt{5}}$

$=40\sqrt{5}+40\sqrt{5}$

$=80\sqrt{5}$ m (05) [15]

(15) (a) Using intergration by parts.

$\int x^2 \sin^{-1} x \, dx = \int \sin^{-1} x \; d\left(\frac{x^3}{3}\right) dx$ (05)

$=\frac{x^3}{3}\sin^{-1}x-\frac{1}{3}\int x^3 \cdot \frac{1}{\sqrt{1-x^2}}dx$ (10)

$=\frac{x^2}{3}\sin^{-1}x+\frac{1}{3}\int x^2 d\left(\sqrt{1-x^2}\right)$ (10)

$=\frac{x^3}{3}\sin^{-1}x+\frac{1}{3}\left[x^2\sqrt{1-x^2}-\frac{1}{3}\int\sqrt{1-x^2}\,2x\,dx\right]$

(05) (05)

$=\frac{-x^3}{3}\sin^{-1}x+\frac{1}{3}x^2\sqrt{1-x^2}+\frac{2}{9}\left(1-x^2\right)^{3/2}+c$, Where (05)

C is an arbitrary constant. [50]

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(b) $\dfrac{x^2+3x+4}{(x^2-1)(x+1)^2}$

$\dfrac{x^2+3x+4}{(x-1)(x+1)^3} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+1)} + \dfrac{C}{(x+1)^2} + \dfrac{D}{(x+1)^3}$ (10)

$x^2+3x+4 = A(x+1)^3 + B(x+1)^2(x-1) + C(x-1)(x+1) + D(x-1)$ (05)

$x = 1, \quad 1+3+4 = 8A \Rightarrow A = 1$
$x = -1 \quad 1-3+4 = -2D \Rightarrow D = -1$

Equating coefficient of x^3; $0 = A + B \Rightarrow B = -A = -1$ (10)

Equating coefficient x^2; $1 = 3A - B + 2B + C \Rightarrow C = -1$

$$\frac{x^2+3x+4}{(x^2-1)(x+1)^2} = \frac{1}{x-1} - \frac{1}{x+1} - \frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}$$

$$\int \frac{x^2+3x+4}{(x^2-1)(x+1)^2}\,dx$$

$$= \int \frac{1}{x-1}dx - \int \frac{1}{x+1}dx - \int \frac{1}{(x+1)^2}dx - \int \frac{1}{(x+1)^3}dx$$ (05)

$$= \underset{(05)}{ln|x-1|} - ln|x+1| + \underset{(05)}{\frac{1}{x+1}} + \frac{1}{2}\,\underset{(05)}{\frac{1}{(x+1)^2}} + C^1$$ (05)

Where C^1 is an arbitrary constant. **50**

(c) $aI + bJ = \left(\int_0^{\pi/2} \dfrac{a^2+b^2+a\cos x + b\sin x}{a^2+b^2+a\cos x+b\sin x}\,dx \right) \to (01)$ (05)

$= \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \dfrac{\pi}{2}$ (05)

$bI - aJ = \int_0^{\pi/2} \dfrac{ab + b\cos x - ab - a\sin x}{a^2+b^2+a\cos x+b\sin x}dx \to (02)$

$= ln\left(a^2+b^2+a\cos x+b\sin x\right)_0^{\pi/2}$ (10)

Since $a^2 + b^2 > 1$ (05)

$= ln\left|a^2+b^2+b\right| - ln\left|a^2+b^2+a\right|$ (10)

$= ln\left|\dfrac{a^2+b^2+b}{a^2+b^2+a}\right|$

$(01)\times a + (02)\times b \Rightarrow I = \dfrac{1}{a^2+b^2}\left\{\dfrac{a\pi}{2} + b\,ln\left(\dfrac{a^2+b^2+b}{a^2+b^2+a}\right)\right\}$ (05)

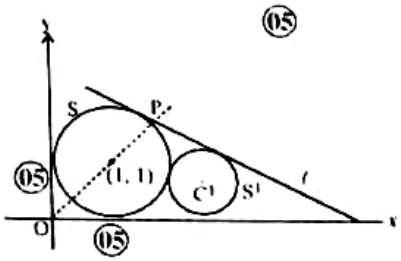
$(01)\times b + (02)\times a \Rightarrow J = \dfrac{1}{a^2+b^2}\left\{\dfrac{b\pi}{2} - a\,ln\left(\dfrac{a^2+b^2+b}{a^2+b^2+a}\right)\right\}$ (05)

50

(16) $x^2 + y^2 - 2x - 2y + 1 = 0$ can be written as

$(x-1)^2 + (y-1)^2 = 1$ (05)

Hence the centre of S $\equiv (1,1)$ and the radius $\underset{(05)}{1}$ (05) **15**



10

Since $P \equiv \left(\underset{(05)}{OP}\cos\frac{\pi}{4}, OP\sin\frac{\pi}{4}\right)$ and $OP = \underset{(05)}{OC} + \underset{(05)}{CP} = \sqrt{2}+1$

We have $P \equiv \left(1+\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$ (05) **20**

Slope of CP = 1 Since $\underset{(05)}{OP} \perp \ell$, Slope of ℓ = -1 (05)

The equation of ℓ is $\underset{(10)}{\left[y-\left(1+\frac{1}{\sqrt{2}}\right)\right] = -1\left[x-\left(1+\frac{1}{\sqrt{2}}\right)\right]}$

$\Rightarrow x + y = 2 + \sqrt{2}$ (05) **25**

The points (0, 0) and (h, k) on the same side of

$\ell : x + y - (2+\sqrt{2}) = 0$ (05)

$\therefore -(2+\sqrt{2})\left[h+k-(2+\sqrt{2})\right] > 0$ (05)

$\Rightarrow h + k < 2 + \sqrt{2}$ (05) **15**

The perpendicular distance from $C^1 \equiv (h,k)$ to ℓ

$d = \dfrac{\left|h+k-(2+\sqrt{2})\right|}{\sqrt{2}}$ (10)

$= \dfrac{(2+\sqrt{2})-(h+k)}{\sqrt{2}}$ (05) $\left(\because h+k < 2+\sqrt{2}\right)$

Since S^1 touches ℓ, this distance must be equal to the radius of S^1 (10)

Since S and S^1 touch each other externally.

$CC^1 = 1 + d \Rightarrow CO^{12} = (1+d)^2$ (10)

$\Rightarrow (h-1)^2 + (k-1)^2 = \left[\dfrac{1+(2+\sqrt{2})-(h+k)}{\sqrt{2}}\right]^2$ (10)

$$\Rightarrow 2h^2+2k^2-4h-4k+4$$

$$=\left[2+2\sqrt{2}-h-k\right]^2$$

$$=\begin{pmatrix}\left(2+2\sqrt{2}\right)^2-2\left(2+2\sqrt{2}\right)h-2\\ \left(2+2\sqrt{2}\right)k+2hk+h^2+k^2\end{pmatrix}$$ (15)

$$\Rightarrow h^2-2hk+k^2+4\sqrt{2}(h+k)=8\left(\sqrt{2}-1\right)$$ (05)

65

(17) (a) $\cos\alpha+\cos\beta-\cos\gamma-\cos(\alpha+\beta+\gamma)$ (05)

$$=\cos\alpha+\cos\beta-\left[\cos\gamma+\cos(\alpha+\beta+\gamma)\right]$$

$$=\begin{bmatrix}2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)-2\cos\\ \left(\frac{\alpha+\beta+2\gamma}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\end{bmatrix}$$ (05) + (05)

$$=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta+2r}{2}\right)$$ (05)

$$=2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot 2\sin\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\beta+\gamma}{2}\right)$$ (05)

$$=4\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha+\gamma)\sin\frac{1}{2}(\beta+\gamma)$$ 25

(b) $f(x)=2\sin^2\frac{x}{2}+2\sqrt{3}\sin\frac{x}{2}\cos\frac{x}{2}+4\cos^2\frac{x}{2}$

$=(1-\cos x)+\sqrt{3}\sin x+2(1+\cos x)$ (10)

$=\sqrt{3}\sin x+\cos x+3$

$=2\left(\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x\right)+3$ (05)

$=2\left(\cos\frac{\pi}{6}\sin x+\sin\frac{\pi}{6}\cos x\right)+3$ (05)

$=2\sin\left(x+\frac{\pi}{6}\right)+3$ (05)

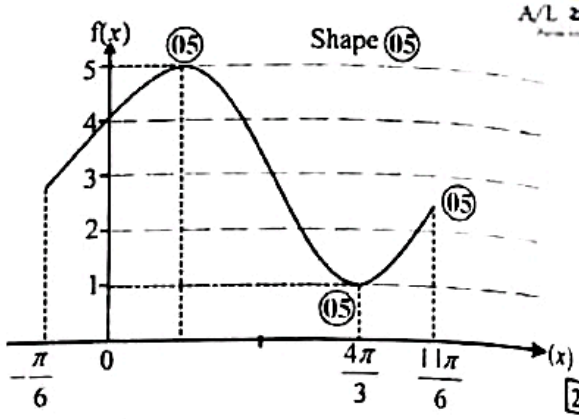
$=A=2,\ B=3,\ \theta=\frac{\pi}{6}$ (10) 35

$-1\le\sin\left(x+\frac{\pi}{6}\right)\le 1$ (05)

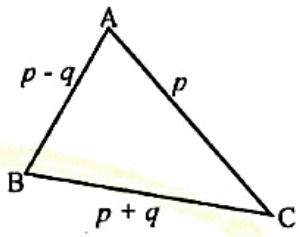
$\Leftrightarrow -2\le 2\sin\left(x+\frac{\pi}{6}\right)\le 2$

$-2+3\le 2\sin\left(x+\frac{\pi}{6}\right)+3\le 2+3$ (05)

$1\le f(x)\le 5$ (05) 15



20



By the sine Rule

$\frac{\sin A}{p+q}=\frac{\sin B}{p}=\frac{\sin C}{p-q}=k$ (05) (05) (05)

$\Rightarrow \sin A-2\sin B+\sin C$

$=k(p+q)-2kp+k(p-q)$

$=0$ (10) 25

$\sin A+\sin C=2\sin B$

$2\sin\frac{A+C}{2}\cos\frac{(A-C)}{2}=2\sin\left(\pi-\left(\overline{A+C}\right)\right)$ (05)

$\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)=\sin(A+C)$ (05)

$\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)=2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A+C}{2}\right)$ (05)

$\Rightarrow\cos\left(\frac{A-C}{2}\right)=2\cos\left(\frac{A+C}{2}\right)$ (05)

$\left(\therefore 0<\frac{A+C}{2}<\frac{\pi}{2},\ \sin\frac{A+C}{2}>0\right)$ 30

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