Combined Mathematics I – 2012 Advanced Level Examination

Part A

01. Using the Principle of Mathematical Induction, prove that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ for any positive integer n.

02. Find the number of arrangements that can be made by using all the letters of the word $ADDING$ find, in how many of these arrangements the vowels are separated.

03. If the coefficient of x and the coefficient of x^2 in the binomial expansion of $(1 + px)^{12}$, where P is a non-zero constant are $-q$ and $11q$ respectively, find the values of p and q.

04. Show that $\lim_{x \to 0} \frac{x \sin x}{2sin^2 3x - x^2 \cos x} = \frac{1}{17}$.

05. Find constant A and B such that $2e^x + 3e^{-x} = A(2e^x - e^{-x}) + B(2e^x + e^{-x})$

Hence, find $\int \frac{2e^x + 3e^{-x}}{2e^x + e^{-x}} dx$.

06. Let l be the straight line passing through the points $(4, 0)$ and $(0, 2)$ and m be the straight line passing through the points $(2, 0)$ and $(0, 3)$. Find the equations of the straight lines l and m. Hence find the equation of the straight line through the origin and the point of intersection of l and m.

07. A curve C is given by the equation $y = 4 - 4x + 3x^2 - x^3$. Find the equation of the tangent drawn to the curve C at the point $(1, 2)$ show that this tangent is perpendicular to the tangent drawn to the curve $y^2 = 4x$ at the point $(1, 2)$. The gradient of the tangent drawn to the curve C at the point $(1, 2)$.

08. Show that the equation of any circle through the points $(2, 0)$ and $(0, 2)$ can be written as $x^2 + y^2 - 4 + \lambda(x + y - 2) = 0$ where λ is a parameter. Find the centre and the radius of this circle in terms of λ.

09. Find the equation of the circle S with AB, where $A \equiv (1, 3)$ and $B \equiv (2, 4)$ as a diameter also find the equation of the circle with centre $(-1, 2)$ which cuts the circle S orthogonally.

10. Taking $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ show that $\tan\left(\frac{\pi}{12}\right) = 2\sqrt{3}$

    Deduce the value of $\tan\left(\frac{23}{12}\pi\right)$.

Part B

❖ Answer five questions only.

11. (a) $f(x) \equiv x^2 + 2kx + k + 2$, where k is a real constant.

    (i) Express $f(x)$ in the form $(x - a)^2 + b$, where a and b are constants to be determined in terms of k.

    Find the turning point of $f(x)$ without using calculus and show that this point is a minimum point.

    Find the minimum value of $f(x)$ in terms of k.

    Hence, show that the curve $y = f(x)$.

    (a) lies entirely above the x-axis if $-1 < k < 2$,

    (b) touches the x-axis if $k = -1$ or $k = 2$,

    (c) cuts the x-axis in two distinct points if $k < -1$ or $k > 2$.

    (ii) Prove that the straight line $y = mx$ intersects the curve $y = f(x)$ in two real and distinct points for all real and finite values of m if and only if $k < -2$.

    (b) Let $g(x) \equiv x^4 + 4x^3 + 7x^2 + 6x + 2$.

    Using Remainder theorem repeatedly show that $(x + 1)^2$ is a factor of $g(x)$.

Express $g(x)$ in the form $(x-a)^2(x^2+bx+c)$, where a, b and c are constants to be determined.

Deduce that $g(x) \geq 0$ for all real values of x.

12. (a) Find constants A and B such that $12x^2+1 \equiv A(2x-1)^3 + B(2x+1)^3$ for all $x \in \mathbb{R}$.

Hence, determine $f(r)$ for $r \in \mathbb{R}^+$, such that $U_r = f(r) - f(r+1)$, where $U_r = \frac{12r^2+1}{(2r-1)^3(2r+1)^3}$.

Show that $\sum_{r=1}^{n} U_r = \frac{1}{2} - \frac{1}{2(2n+1)^3}$

Show that the series $\sum_{r=1}^{\infty} U_r$ is convergent and find the value of $\sum_{r=1}^{\infty} U_r$

(b) Sketch, in the same figure, the graphs of $y = |2x-1|$ and $y = |x| + \frac{5}{3}$.

Hence, find the set of values of x for which $3|x| \geq |6x-3| - 5$.

By considering the graph of $y = |x| - k$, for any $k \in \mathbb{R}$, in the same figure find for what value of l the equation $3|x| = |6x-3| + l$ has only one real solution.

13. (a) Let $A = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$ be a 2×2 matrix.

Show that $\mathbf{A^2 - 3A + 2I = 0}$, where $\mathbf{I}$ is the 2×2 identity matrix and $\mathbf{O}$ is the 2×2 zero matrix.

Hence, find $\mathbf{A}^{-1}$.

Let $B = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ be a 2×2 matrix.

Show that $\mathbf{BA = B}$.

Hence, or otherwise find a non-zero 2×2 matrix $\mathbf{C}$ such that $\mathbf{BC = O}$.

(b) Let z be a complex number.

Prove that $|z|^2 = z\bar{z}$ and $|z| \geq \text{Re}z$

Hence, show that $|z_1| - |z_2| \leq |z_1 - z_2|$ for any two complex numbers z_1 and z_2.

Deduce that $|z_1 + z_2| \leq |z_1| + |z_2|$.

If $|z-i| < \frac{1}{2}$ then, show that $\frac{1}{2} < |z| < \frac{3}{2}$.

Shade the region R consisting the set of points in the Argand diagram which represent the complex number z for which $|z-i| \leq \frac{1}{2}$ and $\frac{\pi}{2} \leq \arg z \leq \frac{2\pi}{3}$.

14. (a) By considering only the first derivative find the minimum and maximum values of $\frac{x^3}{x^4+27}$.

Sketch the graph of $y = \frac{x^3}{x^4+27}$.

Hence, find for what values of k, the equation $kx^4 - x^3 + 27k = 0$, where k is real, has

(i) two coincident real roots,

(ii) three coincident real roots,

(iii) two distinct real roots,

(iv) no real roots.

(b) Consider a rectangle $ABCD$ with $AB = a$ and $BC = b(< a)$. Let P be a movable point on CD. The length of $AP + PB$ is $L(x)$, Where $DP = x$.

Show that $L(x) = \sqrt{x^2 + b^2} + \sqrt{(a-x)^2 + b^2}$.

Find the minimum length of $L(x)$ and the position of P on CD corresponding to this minimum length. Also, find the maximum length of $L(x)$.

15. (a) Show that $\int_0^{\pi} (sin^3 x - cos^3 x)dx = \frac{4}{3}$.

(b) Using integration by parts, or otherwise find $\int x^3 \tan^{-1} x \, dx$.

(c) Using Partial fractions find $\int \frac{2x^2 - 3}{(x-2)^2(x^2+1)} dx$.

16. (a) Find the equations of the bisectors of the angles between two non parallel straight lines $l_1 \equiv a_1x + b_1y + c_1 = 0$ and $l_2 \equiv a_2x + b_2y + c_2 = 0$.

Show that the bisector of the acute angle between two straight lines given by $2x - 11y - 10 = 0$ and $10x + 5y - 2 = 0$ is the bisector of the acute angle between two straight lines given by $4x - 7y - 8 = 0$ and $8x + y - 4 = 0$.

(b) Show that, for all value of g and f the circle $x^2 + y^2 + 2gx + 2fy - r^2 = 0$ bisects the circumference of the circle $x^2 + y^2 - r^2 = 0$.

Show that two circles can be drawn through the point $(1, 1)$ touching the straight line $y + 5 = 0$ and bisecting the circumference of the circle $x^2 + y^2 - 4 = 0$.

Find the equations of these two circles.

17. (a) For a triangle ABC, prove in the usual notation, that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Deduce that $a = (b - c)\cos\frac{A}{2}\text{cosec}\frac{B-C}{2}$.

(b) Show that, for any real value of θ, the expression $\tan\theta - 2\tan\left(\theta - \frac{\pi}{4}\right)$ cannot take any value between -7 and 1.

(c) Express $5cos^2\theta + 18\cos\theta\sin\theta + 29sin^2\theta$ in the form of $a + b\cos(2\theta + \alpha)$, where a and b are constants and α is an angle independent of θ.

Hence or otherwise find the general solution of the equation

$8(\cos x + \sin x)^2 + 2(\cos x + 5\sin x)^2 = 19$.

Maths 4U .com