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G.C.E. (Advanced Level) Examination - August 2012

10 - Combined Mathematics - I

Provisional Scheme of Marking

Part A

01. When n = 1 we have L. H. S = 1 = R. H. S (05)
Thus the result is true for n = 1
Assume that the result is true for n = P

i. e. assume that $1 + 2 + \ldots\ldots + P = \frac{p(p+1)}{2}$ (05)

When n = P + 1 We have

$$1+2+3+\ldots\ldots+p+(p+1)=\frac{p(p+1)}{2}+p+1=\frac{(p+1)\{(P+1)+1\}}{2}$$ (05) (05)

Thus the result is true for n = P + 1
By the Principle of Mathematical Induction,

$1 + 2 + \ldots\ldots + n = \frac{n(n+1)}{2}$ for any positive integern. (05) [25]

02. There are six letters in which two letters are of the same kind.
The number of arrangements that can be made by using all the letters.

$$= \frac{6!}{2!} = 6\times5\times4\times3 = 360$$ (05) (05) [10]

There are two vowels, namely, A and I Taking these two vowels as one letter the number of arranegements with the vowels together that can be made

$$= \frac{5!}{2!} = 5\times4\times3 = 60$$ (05)

Since A and I can be inter change the total number of arrangements with the vowels together that can be made = 2 x 50 = 120 (05)
Thus, the total number of arrange ments with the vowels separated = 360 - 120 = 240 (05) [15]

03. The term in x in the binomial expansion of $(1 + Px)^{12}$ is $^{12}C_1 Px$

The coefficient of x; $^{12}C_1 p = -q \Rightarrow 12p = -q \longrightarrow (1)$ (05)

The term in x^2 in the binomial expansion of $(1 + Px)^{12}$ is $^{12}C_2 P^2x^2$

The coefficient of x^2 : $^{12}C_2 p^2 = 11q \Rightarrow 66p^2 = 11q$

$\Rightarrow 6p^2 = q \rightarrow (2)$ (05)

(1) + (2) $\Rightarrow$ p = -2 (05) or 0

Since P is a non zero constant we have P = -2 From (1) we have q = 24 (05) [25]

04. $$\lim_{x\to0}\frac{x\sin x}{2\sin^2 3x - x^2\cos x} = \lim_{x\to0}\frac{\frac{\sin x}{x}}{2\frac{\sin^2 3x}{x^2}-\cos x}$$

$$= \lim_{x\to0}\frac{\frac{\sin x}{x}}{18\left(\frac{\sin 3x}{3x}\right)^2 - \cos x}$$ (05) (05) (05)

$$= \frac{\lim_{x\to0}\frac{\sin x}{x}}{18\left[\lim_{x\to0}\left(\frac{\sin 3x}{3x}\right)\right]^2 - \lim_{x\to0}\cos x}$$ (05)

$$= \frac{1}{18-1}$$ (05)

$$= \frac{1}{17}$$ [25]

05. $2e^x + 3e^{-x} = A(2e^x - e^{-x}) + B(2e^x + e^{-x})$

$(2A + 2B - 2)e^x + (-A + B - 3)e^{-x} = 0 \Rightarrow A + B = 1$ and

$-A + B = 3$

$\Rightarrow$ A = -1 B = 2 (05) (05) [10]

$$\int\frac{2e^x+3e^{-x}}{2e^x+e^{-x}}dx = \int\frac{-\left(2e^x-e^{-x}\right)+2\left(2e^x+e^{-x}\right)}{2e^x+e^{-x}}dx$$ (05)

$$= \int\frac{-\left(2e^x-e^{-x}\right)}{2e^x+e^{-x}}dx + 2\int dx$$

$$= -\ell n\left|2e^x+e^{-x}\right| + 2x + C$$ (10) [15]

06. The equatien of l is $\frac{y}{x-4} = \frac{2-0}{0-4} \Rightarrow 2x + 4y - 8 = 0$ (05)
$\Rightarrow x + 2y - 4 = 0$ [05]

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The equation of m is $\frac{y}{x-2} = \frac{3-0}{0-2} \Rightarrow 3x + 2y - 6 = 0$ (05) [05]

Any straight line through the point of intersection of l and m can be written as $x + 2y - 4 + \lambda(3x + 2y - 6) = 0$, where λ is a parameter. (05)

Since this line passes through the origin we have

$\lambda = -\frac{2}{3}$ (05)

Thus the required equtions is $2y - 3x = 0$ (05) [15]

06. Aliter

Any straight line through the origin can be written as $y - \mu x = 0$. Where μ is a para meter. (05)

The Point of intersection of l and m is $\left(1, \frac{3}{2}\right)$ (05)

Since the line passes through this point we have

$\mu = \frac{3}{2}$

Thus the required equation is $2y - 3x = 0$ (05) [15]

07. The gradient of the tangent drawn to the curve C at the point (1, 2)

$$= \frac{dy}{dx}\Big/_{(1,2)} = \underset{(05)}{\left(-4 + 6x - 3x^2\right)\Big/_{x=1}} = -1 \ (05)$$

Thus the equation of the tangent to the curve C at the point (1, 2) is $\frac{y-2}{x-1} = -1$

i. e. $x + y - 3 = 0$ (05) [15]

The gradient of the tangent drawn to the parabola at the point (1, 2)

$$= \frac{dy}{dx}\Big/_{(1,2)} = \frac{2}{y}\Big/_{y=2} = 1 \ (05)$$

Thus the tangent drawn to the curve C at the point (1, 2) is Perpendicular to the tangent drawn to the curve at the point (1, 2) (05) [10]

08. Let $x^2 + y^2 + 2gx + 2fy + C = 0$ be the equation of any circle through the points (2, 0) and (0, 2)

Then we have $4 + 4g + c = 0$ and $4 + 4f + c = 0$ (05)

From the above two equation we have $f = g$ and $C = -4(g + 1)$ (05)

thus the equation can be written as

$x^2 + y^2 - 2gx + 2gy + 4(g + 1) = 0$

i. e. $x^2 + y^2 - 4 + \lambda(x + y - 2) = 0$ where $\lambda = 2g$ (05) [15]

$$\left(x + \frac{\lambda}{2}\right)^2 + \left(y + \frac{\lambda}{2}\right)^2 - 4 - 2\lambda - \frac{1}{2}\lambda^2 = 0$$

i. e. $\left(x + \frac{\lambda}{2}\right)^2 + \left(y + \frac{\lambda}{2}\right)^2 - \frac{(\lambda + 2)^2 + 4}{2} = 0$

The centre of this circle is $\left[-\frac{\lambda}{2}, -\frac{\lambda}{2}\right]$ (05) [05]

The radius of the circle is i.e $= \sqrt{\frac{(\lambda + 2)^2 + 4}{2}}$ (05) [05]

09. Let (x, y) be any point on the circle S

then we have $\left(\frac{y-3}{x-1}\right)\left(\frac{y-4}{x-2}\right) = -1$ (10)

i. e $x^2 + y^2 - 3x - 7y + 14 = 0$ (05) [15]

Aliter

Let $(-g, -f)$ be the centre of the circle S

Then we have $-g = \frac{3}{2}$ and $-f = \frac{7}{2}$ (05)

The radius of the circle S is

$\sqrt{\frac{(2-1)^2 + (4-3)^2}{2}} = \frac{\sqrt{2}}{2}$ (05)

Thus the eqiation of the circle S is

$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{1}{2}$

i.e. $x^2 + y^2 - 3x - 7y + 14 = 0$ (05) [15]

Let the equation of the circle with centre (-1, 2) be

$x^2 + y^2 + 2x - 4y + k = 0$ (05)

Since this circle and the circle S intersect or tho gonally we have.

$2\left(\frac{-3}{2}\right)(1) + 2\left(-\frac{7}{2}\right)(-2) = k + 14 \Rightarrow k = -3$ (05)

Thus the equation of the circle is [10]

$x^2 + y^2 + 2x - 4y - 3 = 0$

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10 $\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}}$ (05)

$= \frac{\sqrt{3}-1}{1+\sqrt{3}}$ (05)

$= \frac{(\sqrt{3}-1)^2}{3-1} = 2-\sqrt{3}$ (05) [15]

$\tan\frac{23\pi}{12} = \tan\left(2\pi - \frac{\pi}{12}\right) = -\tan\frac{\pi}{12}$ (05)

$= -(2-\sqrt{3}) = \sqrt{3}-2$ (05) [10]

Combined Mathematics - I
Part - B

11. (a) $f(x) = x^2 + 2x + k + 2$

$= (x+a)^2 + b$ where $a = 1$, (05)

(05) and $b = 2 + k - k^2$ (05) [15]

The function f(x) decrases from ∞ to the value $2 + k - k^2$ as x increases from $-\infty$ to $-k$ and then increases from $2 + k - k^2$ to ∞ as x increases from $-k$ to ∞ (05)

Thus there is only one turning Point at $x = -k$ and it is a minimum point (05) (05) [15]

The minimum value of $f(x)$ is $2 + k - k^2$ (05) [05]

(i) We can write $2 + k - k^2 = (2 - k)(1+k)$

(α) $-1 < k < 2$ then, the minimum value of $f(x)$ is (05) Positive and hence the entire curve should lie above the x - axis. (05) [10]

(β) If $k = -1$ or $k = 2$ then, the minimum value of (05) $f(x)$ is zero and hence the curve should touch the x - axis (05) [10]

(γ) If $k < -1$ or $k > 2$ then, the minimum value of $f(x)$ is negative and hence the curve should out the x - axis in two distinct points. (05) (05) [10]

(ii) If the straight line $y = mx$ intersects the curve $y = f(x)$ then the abscissa of any point of intersection should satisfy the equation.

$x^2 + 2kx + k + 2 = mx$ (05)

i.e. $x^2 + (2k - m)x + k + 2 = 0$ (05)

Thus, the straight line $y = mx$ intersects the curve $y = f(x)$ in two real and distinct points if and only if $x^2 + (2k - m)x + k + 2 = 0$ has two distinct roots

i.e. if and only if $(2k - m)^2 - 4(k + 2) > 0$ (05)

For all real and finite values of m, $(2k - m)^2 - 4(k + 2) > 0$ if and only if $k > -2$ (05)

Thus, the straight line $y = mx$ intersects the curve $y = f(x)$ in two real and distinct points for all real and finite values of m if and only if $k > -2$ (05) [25]

(b) $f(-1) = (-1)^4 + 4(-1)^3 + 7(-1)^2 + 6(-1) + 2 = 0$ (05)

The remainder when f(x) is divided by $(x + 1)$ is 0

$\therefore (x + 1)$ is a factor of $f(x)$ (05)

Thus $f(x) = (x + 1)Q(x)$. (05) Where $Q(x)$ is the quotient, when f(x) is divided by $(x + 1)$

$Q(x) = x^3 + 3x^2 + 4x + 2$ (05)

$Q(-1) = (-1)^3 + 3(-1)^2 + 4(-1) + 2 = 0$ (05)

The remainder, when Q(x) is divided by $(x + 1)$ is 0

Thus, $Q(x) = (x + 1)R(x)$ where R (x) is the quotient when (05)

Q(x) is divided by $(x + 1)$

$\therefore (x + 1)$ is a factor of $Q(x)$ (05)

$\therefore (x + 1)^2$ is a factor of $f(x)$ [35]

$R(x) = x^2 + 2x + 2$ (05)

$g(x) = (x + 1)^2(x^2+2x+2)$ (10)

$g(x) = (x-a)^2(x^2+bx+c)$, where $a = -1$, $b=2$, and $c = 2$ [15]

$f(x) = (x+1)^2\left(x^2 + 2x + 2\right) = (x+1)^2\left\{(x+1)^2 + 1\right\} \geq 0$ for all real (05) values of x. (05) [10]

12.(a) $12r^2 + 1 \equiv A(2r - 1)^3 + B(2r + 1)^3$ for all $r \in R$

For $r = \frac{1}{2}$ we have $B = \frac{1}{2}$ (05)

for $r = -\frac{1}{2}$ we have $A = -\frac{1}{2}$ (05) [10]

$\frac{12r^2 + 1}{(2r - 1)^3(2r + 1)^3} = \frac{-\frac{1}{2}(2r - 1)^3 + \frac{1}{2}(2r + 1)^3}{(2r - 1)^3(2r + 1)^3}$ (05)

$= \frac{1}{2(2r - 1)^3} - \frac{1}{2(2r + 1)^3}$ (05)

$= f(r) - f(r + 1)$ where

$= f(r) = \frac{1}{2(2r-1)^3}$ (05) [15]

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$u_1 = f(1) - f(2)$ (05)

$u_2 = f(2) - f(3)$ (05)

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$u_{n-1} = f(n-1) - f(n)$ (05)

$u_n = f(n) - f(n+1)$ (05)

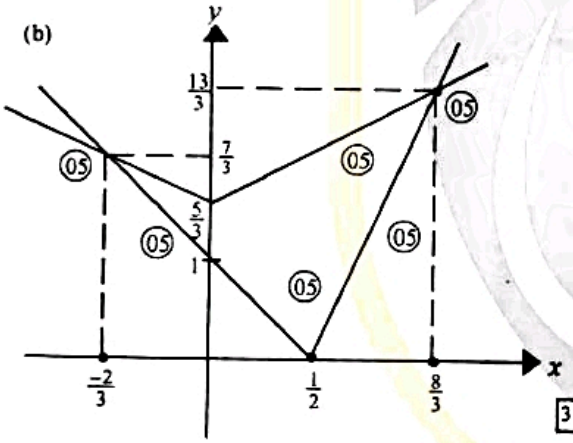
$$\sum_{r=1}^{n} u_r = f(1) - f(n+1) = \frac{1}{2} - \frac{1}{2(2n+1)^3}$$ (05) (05) 30

$$\sum_{r=1}^{\infty} u_r = \lim_{n\to\infty} \sum_{r=1}^{n} u_r = \lim_{n\to\infty} \left\{ \frac{1}{2} - \frac{1}{2(2n+1)^3} \right\} = \frac{1}{2}$$ which is finite. (05)

Thus, the series $\sum_{r=1}^{\infty} u_r$ is con vergent (05) 10

The value of $\sum_{r=1}^{\infty} u_r$ is $\frac{1}{2}$ (05) 05

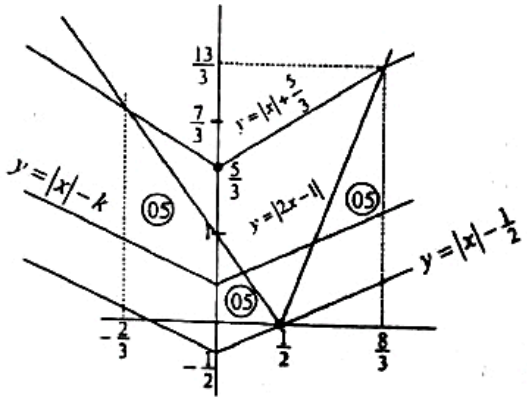
(b)



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$3|x| \geq |6x-3| - 5 \Rightarrow |x| + \frac{5}{3} \geq |2x-1|$ (05)

Thus the set of values of x which $3|x| \geq |6x-3| - 5$ is $\left\{ x : -\frac{2}{3} \leq x \leq \frac{8}{3} \right\}$ (05) 10



15

$3|x| = |6x-3| + \ell \Rightarrow |x| - \frac{\ell}{3} = |2x-1|$ has only one real root when the graph of $y = |x| + k$ Passes (05) through the Point $\left(\frac{1}{2}, 0\right)$ (05)

For this to happen $k = \frac{1}{2}$ (05)

$|x| - \frac{\ell}{3} = |x| - \frac{1}{2} \Rightarrow \ell = \frac{3}{2}$ (05) (05) 25

13. (a) $A^2 - 3A + 2I = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix} - 3\begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (05)

$= \begin{pmatrix} 10 & 9 \\ -6 & -5 \end{pmatrix} - \begin{pmatrix} 12 & 9 \\ -6 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

(05) (05) (05) 20

$2A^{-1} = 3AA^{-1} - AAA^{-1} = 3I - A$ (05)

$= 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix}$ (05)

$A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix}$ 10

$BA = \begin{pmatrix} 2, & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = B$ (05) 05

$BA - B = 0 \Rightarrow B(A - I) = 0 \Rightarrow BC = 0$ Where (05)

$C = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -2 & -2 \end{pmatrix}$ (05) 15

(b) Let $z = x + iy$ where $x, y, \in \mathbb{R}$

$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = \left\{\sqrt{x^2+y^2}\right\}^2 = |z|^2$ (05) 05

$|z| = \sqrt{x^2+y^2} \geq \sqrt{x^2} = \sqrt{|x|^2} = |x| \geq x = R_e z$ (05) 10

$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2})$

$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$ (05)

$= z_1\bar{z}_1 + z_2\bar{z}_2 - z_2\bar{z}_1 - z_1\bar{z}_2$

$= z_1\bar{z}_1 + z_2\bar{z}_2 - 2Re(z_1\bar{z}_2)$

$= |z_1|^2 + |z_2|^2 - 2Re(z_1\bar{z}_2)$ (05)

$\geq |z_1|^2 + |z_2|^2 - 2(z_1\bar{z}_2)$ (05)

$= |z_1|^2 + |z_2|^2 - 2|z_1||\bar{z}_2|$

$\geq (|z_1| - |z_2|)^2 - 2|z_1| \cdot |z_2|$ (05)

$= (|z_1| - |z_2|)^2$ (05) 25

$|z_1| - |z_2| \leq |z_1 - z_2|$

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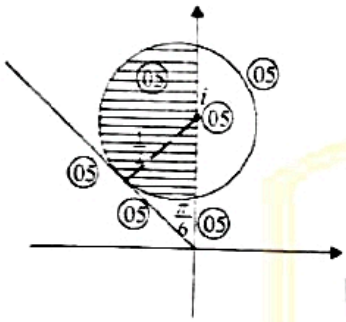
$|z_1| = |(z_1 + z_2) - z_2| \geq |z_1 + z_2| - |z_2|$ (05) (05)

i.e. $|z_1 + z_2| \leq |z_1| + |z_2|$ 10

$|z| = |i| \leq |z - i| < \frac{1}{2} \Rightarrow |z| - 1 < \frac{1}{2} \Rightarrow |z| < \frac{3}{2}$ (05) (05)

$|i| - |z| \leq |i - z| = |z - i| < \frac{1}{2} \Rightarrow 1 - |z| < \frac{1}{2} \Rightarrow |z| > \frac{1}{2}$ (05) (05)

i.e. $\frac{1}{2} < |z| < \frac{3}{2}$ 20



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14. (a) Let $f(x) = \frac{x^3}{x^4 + 27}$

Then

$f^1(x) = \frac{3x^2(x^4 + 27) - x^3(4x^3)}{(x^4 + 27)^2}$ (05)

$= \frac{x^2(81 - x^4)}{(x^4 + 27)^2} = 0$ (05)

$\Rightarrow x = -3, 0, 3$ (05)

x	$(-\infty, -3)$	(-3, 0)	(0, 3)	$(3, \infty)$
$f(x)$	-	+	+	-

(05) (05)

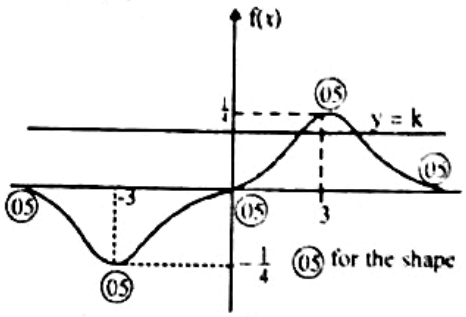
Thus $f(x)$ has a minimum at x = -3 and a maximum at x = 3 (05)

$f(-3) = \frac{(-3)^3}{(-3)^4 + 27} = -\frac{1}{4}$ and $f(3) = \frac{3^3}{3^4 + 27} = \frac{1}{4}$ (05) (05)

Thus the minimum value and maximum value of $f(x)$ are $-\frac{1}{4}$ and $\frac{1}{4}$

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$f(x) \to \pm 0$ as $x \to \pm\infty$



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(05) Chsider y = k and $y = \frac{x^3}{x^4 + 27}$

The abscissas of the points of intersectians of

y = k and $y = \frac{x^3}{x^4 + 27}$ are the solutions of the equation $kx^4 - x^3 + 27k = 0$ (05)

Thus,

(i) When $k = -\frac{1}{4}$ or $\frac{1}{4}$, the line y = k touches the curve $y = \frac{x^3}{x^4 + 27}$ and hence k $x^4 - x^3 + 27k = 0$ has two coincident real roots. (05)

(ii) When k = 0, the line y = k Passes through the curve $y = \frac{x^3}{x^4 + 27}$ and the equation $kx^4 - x^3 + 27k = 0$ reduces to $x^3 = 0$ and it has three. coincident real roots. (05)

(iii) When $-\frac{1}{4} < k < 0$ or $0 < k < \frac{1}{4}$, the line y = k cuts the curve $y = \frac{x^3}{x^4 + 27}$ at two distinct points and hence $kx^4 - x^3 + 27k = 0$ has two distinct real roots. (05)

(iv) when $k < -\frac{1}{4}$ or $k > \frac{1}{4}$ the line y= k does not cuts the curve $y = \frac{x^3}{x^4 + 27}$ and hence $kx^4 - x^3 + 27k$ = 0 has no real roots. (05)

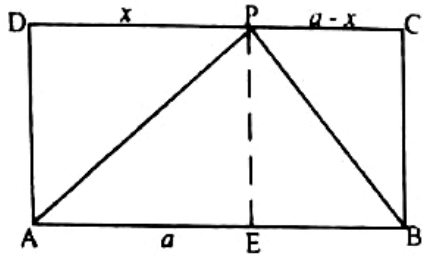
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(b) L(x) = AP + PB (05)

$= \sqrt{AE^2 + EP^2} + \sqrt{BE^2 + EP^2}$

$= \sqrt{x^2 + b^2} + \sqrt{(a - x)^2 - b^2}$ (05)

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$$\frac{dl(x)}{dx} = \frac{x}{\sqrt{x^2+b^2}} - \frac{a-x}{\sqrt{(a-x)^2+b^2}} \quad (05)$$

$$= \frac{x\sqrt{(a-x)^2+b^2} - (a-x)\sqrt{x^2+b^2}}{\sqrt{x^2+b^2}\sqrt{(a-x)^2+b^2}}$$

$$= 0 \Rightarrow x\sqrt{(a-x)^2+b^2} - (a-x)\sqrt{x^2+b^2} = 0 \quad (05)$$

$$\left[\Rightarrow x^2\left[(a-x)^2+b^2\right] = (a-x)^2\left\{x^2+b^2\right\} \Rightarrow x^2 = (a-x)^2 \Rightarrow x = \frac{a}{2}\right]$$

Range of x	$0 \le x < \frac{a}{2}$	$x = \frac{a}{2}$	$\frac{a}{2} < x \le a$
$L^1(x)$	-	(05) 0	+

$L(x)$ is minimum when $x = \frac{a}{2}$ (05)

The minimun length

$$= \sqrt{\left(\frac{a}{2}\right)^2+b^2} + \underset{(05)}{\sqrt{\left(a-\frac{a}{2}\right)^2+b^2}} = \underset{(05)}{\sqrt{a^2+4b^2}}$$

This occurs when P is the midpoint of CD (05)

35

L(x) is maximum when P is at D or C and in this case

$$L(x) = b + \sqrt{a^2+b^2} \quad (05)$$

05

15. (a) $\int_0^{\pi}\left(\sin^3 x - \cos^3 x\right) dx$

$$= \int_0^{\pi}\left\{\left(1-\cos^2 x\right)\sin x - \left(1-\sin^2 x\right)\cos x\right\} dx \quad (05)$$

$$= \Big[\underset{(05)}{-\cos x} + \underset{(05)}{\frac{\cos^3 x}{3}} \underset{(05)}{-\sin x} + \underset{(05)}{\frac{\sin^3 x}{3}}\Big]_0^{\pi}$$

$$= 2 - \frac{2}{3} = \frac{4}{3} \quad (05)$$

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Alter

(a) $\int_0^{\pi}\left(\sin^3 x - \cos^3 x\right) dx$

$$= \int_0^{\pi}\left\{(\sin x - \cos x)\left(\sin^2 x + \sin x\cos x + \cos^2 x\right)\right\} dx \quad (05)$$

$$= \int_0^{\pi}(\sin x - \cos x)(1 + \sin x\cos x)\, dx$$

$$= \int_0^{\pi}\left(\sin x + \sin^2 x\cos x - \cos x - \cos^2 x\sin x\right) dx$$

$$= \Big[\underset{(05)}{-\cos x} + \underset{(05)}{\frac{\sin^3 x}{3}} \underset{(05)}{-\sin x} + \underset{(05)}{\frac{\cos^3 x}{3}}\Big]_0^{\pi}$$

$$= 2 - \frac{2}{3} = \frac{4}{3} \quad (05)$$

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(b) $\int x^3 \tan^{-1} x\, dx$

$$= \frac{x^4}{4}\underset{(05)}{\tan^{-1} x} - \frac{1}{4}\int \frac{x^4}{1+x^2} dx \quad (05)$$

$$= \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\int \frac{\left(x^2+1\right)^2 - \left(2x^2+1\right)}{1+x^2} dx \quad (10)$$

$$= \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\int\left(x^2+1\right)dx + \frac{1}{4}\underset{(10)}{\int \frac{2\left(x^2+1\right)-1}{1+x^2} dx}$$

$$= \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\underset{(05)}{\left(\frac{x^3}{3}+x\right)} + \frac{1}{2}\int dx - \underset{(05)}{\frac{1}{4}\int \frac{dx}{1+x^2}}$$

$$= \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\left(\frac{x^3}{3}+x\right) + \underset{(10)}{\frac{1}{2}x} - \frac{1}{4}\tan^{-1} x + C;$$

50

Where C is an arbitrary constant

(C) $$\frac{2x^2-3}{(x-2)^2\left(x^2+1\right)} = \underset{(05)}{\frac{A}{x-2}} + \underset{(05)}{\frac{B}{(x-2)^2}} + \underset{(05)}{\frac{Cx+D}{x^2+1}}$$

where A, B, C, and D are constant to be determined

$2x^2 - 3 = A(x-2)(x^2+1) + B(x^2+1) + (Cx+D)(x-2)^2$

Taking $x = 2$ we have $B = 1$ (05)

Comparing constant terms we have $-3 = -2A + B + 4D$

Comparing the coefficients of the terms in x^3 we have

$0 = A + C$ (10)

Comparing the coefficients of the terms is x we have

$0 = A + 4C - 4D$

Substituting B=1 in $-3=-2A+B+4D$ we have

$-2 = -A + 2D \rightarrow (01)$

Substituting C=A in $0=A+4C-4D$ we have

$0 = 3A + 4D \rightarrow (02)$

From (01) and (02) we have $D = \underset{(05)}{-\frac{3}{5}}$ and $A = \underset{(05)}{+\frac{4}{5}}$ $C = \underset{(05)}{-\frac{4}{5}}$

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$$\frac{2x^3-3}{(x-2)^2(x^2+1)} = \frac{4}{5(x-2)} + \frac{1}{(x-2)^2} - \frac{4x+3}{5(x^2+1)}$$

$$\int \frac{2x^3-3}{(x-2)^2(x^2+1)}\,dx = \frac{4}{5}\int \frac{dx}{x-2} + \int \frac{dx}{(x-2)^2} - \frac{1}{5}\int \frac{4x+3}{x^2+1}\,dx \quad (05)$$

$$= \frac{4}{5}\ell n\,|x-2| - \frac{1}{x-2} - \frac{2}{5}\int \frac{2x}{x^2+1}\,dx - \frac{3}{5}\int \frac{dx}{x^2+1} \quad (05)\ (05)$$

$$= \frac{4}{5}\ell n\,|x-2| - \frac{1}{x-2} - \frac{2}{5}\ell n\left(x^2+1\right) - \frac{3}{5}\tan^{-1}x + k \quad (10)$$

Where k is an arbitrary constant [10]

16. (a) Let $P_0(x_0, y_0)$ be a Point at the same distance from the two lines

The Perpendicular distances from the point P_0 to the lines $l_1 = 0$ and $l_2 = 0$ are

$$\left|\frac{|a_1x_0 + b_1y_0 + c_1|}{\sqrt{a_1^2+b_1^2}}\right| \quad (05) \qquad \text{and} \qquad \left|\frac{|a_2x_0 + b_2y_0 + c_2|}{\sqrt{a_2^2+b_2^2}}\right| \quad (05)$$ respectively.

Since the two distances are the same we have

$$\left|\frac{|a_1x_0 + b_1y_0 + c_1|}{\sqrt{a_1^2+b_1^2}}\right| = \left|\frac{|a_2x_0 + b_2y_0 + c_2|}{\sqrt{a_2^2+b_2^2}}\right| \quad (05)$$

If the absolute values of two numbers n and m are the same, then either n = m or n = -m using the above fact we have

$$\frac{a_1x_0 + b_1y_0 + c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x_0 + b_2y_0 + c_2}{\sqrt{a_2^2+b_2^2}}$$

Thus the coordinates of any point at equidistanee from the two lines $l_1 = 0$ and $l_2 = 0$ satisfy the equation

$$\frac{a_1x_0 + b_1y_0 + c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x_0 + b_2y_0 + c_2}{\sqrt{a_2^2+b_2^2}} \quad (05)$$

If a Point P(x, y) varies such that the distance from the point P to the lines $l_1 = 0$ and $l_2 = 0$ are the same. then P is a variable Point on a bisecter of an angle between the lines $l_1 = 0$ and $l_2 = 0$ Thus the equations of the bisectors of the angle are.

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2+b_2^2}} \quad (05) \qquad [25]$$

The equations of the bisectors of the angles between the two lines 2x - 11y - 10 = 0 and 10x + 5y - 2 = 0

$$\frac{2x-11y-10}{\sqrt{2^2+(-11)^2}} = \pm\frac{10x+5y-2}{\sqrt{10^2+5^2}} \quad (05)$$

i.e. $2x - 11y - 10 = \pm(10x + 5y - 2) \Rightarrow x + 2y + 1 = 0$ or $2x - y - 2 = 0$ (05) (05)

Let θ be the angle be tween the line 2x - 11y - 10 = 0 and x + 2y + 1 = 0

Then $\tan\theta = \left|\frac{\frac{2}{11} - \left(-\frac{1}{2}\right)}{1 + \frac{2}{11}\left(-\frac{1}{2}\right)}\right| = \frac{3}{4} < 1$ (05)

Thus the equation of the bisector of the acute angle between two straight lines given by 2x - 11y - 10 = 0 and 10x + 5y - 2 = 0 is x + 2y + 1 = 0 (05)

The equations of the bisectors of the angles between the two lines 4x - 7y - 8 = 0 and 8x + y - 4 = 0 are

$$\frac{4x-7y-8}{\sqrt{4^2+(-7)^2}} = \pm\frac{8x+y-4}{\sqrt{8^2+1^2}} \quad (05)$$

i.e. $4x - 7y - 8 = \pm(8x + y - 4) \Rightarrow x + 2y + 1 = 0$ and $2x - y - 2 = 0$ (05) (05)

Let ϕ be the angle between the line 4x - 7y - 8 =0 and x + 2y + 1 = 0

Then $\tan\phi = \left|\frac{\frac{4}{7} - \left(-\frac{1}{2}\right)}{1 + \left(\frac{4}{7}\right)\left(-\frac{1}{2}\right)}\right| = \frac{3}{2} > 1$ (05)

Thus the equation of the bisector of the obtuse angle between two straight lines given by 4x - 7y - 8 = 0 and 8x + y - 4 = 0 is x+ 2y+ 1 = 0 (05) [50]

(b) Let $P_0(x_0, y_0)$ be a point of inter section of the two circles $x^2 + y^2 + 2gx + 2fy - r^2 = 0$ and $x^2 + y^2 - r^2 = 0$

Then we have $x_0^2 + y_0^2 + 2gx_0 + 2fy_0 - r^2 = 0 \rightarrow (01)$

$$x_0^2 + y_0^2 - r^2 = 0 \rightarrow (02)$$

$$(1)-(2) \Rightarrow 2gx_0 + 2fy_0 \Rightarrow gx_0 + fy_0 = 0 \quad (05)$$

Thus the points of intersection lie on the line gx + fy = 0 (05)

Since this line passes through the origin and origin is the centre of the cirele.

$X^2 + y^2 - r^2 = 0$ for all values of g and f the circle $X^2 + y^2 + 2gx + 2fy - r^2 = 0$ bisects the circumference of the circle $x^2 + y^2 - r^2 = 0$ (05) [15]

For allvalues of g and f the circle $x^2 + y^2 + 2gx + 2fy - 4 = 0$ bisects the circumference of the circle $x^2 + y^2 - 4 = 0$ (05)

Radius of the circle $x^2 + y^2 + 2gx + 2fy - 4 = 0$ is $\sqrt{g^2+f^2+4}$ (05)

The distance from the centre (-g - f) to the line y + 5 = 0 is $|-f+5|$ (05)

Since the circle $x^2 + y^2 + 2gx + 2fy - 4 = 0$ touches the line y + 5 = 0 we have

$$\sqrt{g^2+f^2+4} = |-f+5| \quad (05) \Rightarrow g^2 + f^2 + 4 = f^2 - 10f + 25$$

$$\Rightarrow g^2 + 10f - 21 = 0 \rightarrow (01)$$

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If the circle $x^2 + y^2 + 2gx + 2fy - 4 = 0$ Passes through the Point (1,1) we have $1 + 1 + 2g + 2f - 4 = 0 \Rightarrow g + f = 1$ (05)

Substituting $g + f = 1$ in (1) we have

$g^2 - 10g - 11 = 0 \Rightarrow (g - 11)(g + 1) = 0 \Rightarrow g = 11$ or -1 (05) (05)

When $g = 11$ we have $f = -10$ (05)

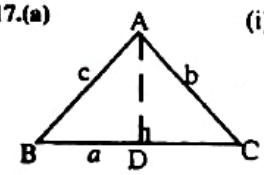
When $g = -1$ we have $f = +2$ (05)

Since we have two sets of values for g and f there are two circles. (05) [5.0]

The equations of the two circles are.

$x^2 + y^2 + 22x - 20y - 4 = 0$ and $x^2 + y^2 - 2x + 4y - 4 = 0$ (05) (05) [10]

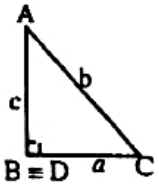
17.(a)



(i) ABC is an acute angled triangle.

$AD = AB \sin B = AC \sin C$

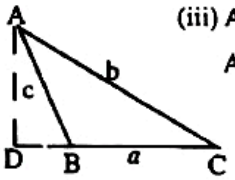
$\therefore \frac{b}{\sin B} = \frac{c}{\sin C}$ (05)



(ii) ABC is an right angled triangle

$AD = AB\sin B = AC\sin C$

$\therefore \frac{b}{\sin B} = \frac{c}{\sin C}$ (05)



(iii) ABC is an obtuse angled triangle.

$AD = AB\sin(\pi - B) = AC\sin C$

$\therefore \frac{b}{\sin B} = \frac{c}{\sin C}$ (05)

Similarly we can show $\frac{a}{\sin A} = \frac{c}{\sin C}$ (05)

$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

[20]

$$\frac{a}{\sin A} = \frac{b-c}{\sin B - \sin C}$$ (05)

$$\frac{a}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{b-c}{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}$$ (05) (05)

$$= \frac{b-c}{2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}$$ (05)

$$a = (b-c)\cos\frac{A}{2}\,cosec\frac{B-C}{2}$$ (05)

[25]

(b) Let $k = \tan\theta - 2\tan\left(\theta - \frac{\pi}{4}\right)$

$$= \tan\theta - 2\frac{\tan\theta - 1}{1 + \tan\theta} = \frac{\tan^2\theta - \tan\theta + 2}{1 + \tan\theta}$$ (05)

i.e. $\tan^2\theta - (k+1)\tan\theta + 2 - k = 0$

This equation has real solutions if and only if $(k+1)^2 - 4(2-k) \geq 0$ (05) (05)

i.e. if and only if $k^2 + 6k - 7 \geq 0$ (05)

i.e. if and only if $(k+7)(k-1) \geq 0$ (05)

i.e. and only if $k \leq -7$ or $k \geq 1$ (05)

Thus for any real value of θ, the expression

$\tan\theta - 2\tan\left(\theta - \frac{\pi}{4}\right)$ cannot take any value between -7 and 1 [30]

(c) $5\cos^2\theta + 18\cos\theta\sin\theta + 29\sin^2\theta$ (05)

$$= \frac{5}{2}(1 + \cos 2\theta) + 9\sin 2\theta + \frac{29}{2}(1 - \cos 2\theta)$$

$$= 17 - 12\cos 2\theta + 9\sin 2\theta$$ (05)

$$= 17 - 15\left(\frac{4}{5}\cos 2\theta - \frac{3}{5}\sin 2\theta\right)$$ (05)

$= 17 - 15(\cos\alpha\cos 2\theta - \sin\alpha\sin 2\theta)$, where $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$ (05)

$= 17 - 15\cos(2\theta + \alpha)$; where $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$ (05)

$= a + b\cos(2\theta + \alpha)$, where $a = 17$, $b = -15$ and α is such that

$\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$ [25]

$$8(\cos x + \sin x)^2 + 2(\cos x + 5\sin x)^2 = 19$$

$$8(\cos^2 x + 2\cos x\sin x + \sin^2 x) + 2(\cos^2 x + 10\cos x\sin x + 25\sin^2 x) = 19$$ (05) (05)

$$10\cos^2 x + 36\cos x\sin x + 58\sin^2 x = 19$$ (05)

$$5\cos^2 x + 18\cos x\sin x + 29\sin^2 x = \frac{19}{2}$$ (05)

$17 - 15\cos(2x + \alpha) = \frac{19}{2}$, α is such that $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$ (05)

$$15\cos(2x + \alpha) = 17 - \frac{19}{2} = \frac{15}{2}$$ (05)

$$\cos(2x + \alpha) = \frac{1}{2} = \cos\frac{\pi}{3}$$

$2x + \alpha = 2n\pi \pm \frac{\pi}{3}$; where n is an integer (05)

$x = n\pi - \frac{\alpha}{2} \pm \frac{\pi}{6}$; where n is an integer and

α is such that $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$ (05)

[50]