# Combined Mathematics II – 2012 Advanced Level Examination

## Part A

1. A boy running due south along a straight road with speed $u\ kmh^{-1}$, feels a wind blowing due west. When he is running due north along a straight road with the same speed. He feels the wind blowing due south-west. Draw the velocity triangles of relative velocities for the motions of the wind in the same figure. Hence find the true speed and the direction of the wind.

2. A particle of mass $m$ is released from rest at the top of a slope along the line of greatest slope which is inclined at an angle $\alpha$ to the horizontal. It the particle takes one second to move down a distance $d$ from the top. Assuming that the resistance $R$ to the motion of the particle is constant. Show that $R = m(g\sin\alpha - 2d)$. Also, find the velocity of the particle when the distance travelled is $d$ from the top.

3. A smooth particle of mass $m$, which is at a height $h$ above a horizontal smooth plane. Falls under gravity from rest and strikes the plane and then rebounds. If the loss of kinetic energy due to the impact is $\frac{mgh}{4}$. Find the coefficient of restitution between the particle and the plane show that the particle rebounds to a height $\frac{3h}{4}$.

4. A particle $P$ of mass $m$ is attached to one end of a light inextensible string of length $l$ and the other end of the string is attached to a fixed point $O$. When the particle $P$ is hanged freely in a vertical plane it is given a velocity $2gl$ in the vertical plane perpendicular to $OP$. Using the principle of conservation of Energy, find the velocity of the particle $P$ when $OP$ makes an angle $\frac{\pi}{3}$ with the down ward vertical. Show that the tension of the string at this instant is $\frac{3}{2}mg$.

5. $\underline{a} = \underline{i}\sqrt{3}\underline{j}$ where $\underline{i}$ and $\underline{j}$ have the usual meaning $\underline{b}$ is a vector with magnitude $\sqrt{3}$. If the angle between the vectors $\underline{a}$ and $\underline{b}$ is $\frac{\pi}{3}$ . Find $\underline{b}$ in the form $x\underline{i} + y\underline{i}$ where $x(< 0)$ and $y$ are constants to be determined.

6. A uniform rod $AB$ of weight $W$ and length $2a$ is in equilibrium with the end $A$ on a rough horizontal ground and the end $B$ against a smooth vertical wall which is perpendicular to the vertical plane containing the rod $AB$. If the coefficient of friction between the rod and the ground is $\sqrt{\frac{3}{2}}$ . Find the inclination of the rod to the horizontal when it is just about to slip.

7. Let $A, B$ and $C$ be mutually exclusive and exhaustive events of a sample space $\Omega$ if $P(A) = 2p, P(B) = p^2$ and $P(C) = 4p - 1$, find the value of $p$.

8. Let $A, B$ and $C$ be three independent events of a sample space $\Omega$. Show that $A$ and $B \cup C$ are independent events.

9. The mean and the standard deviation of 100 observations were calculated as 30 and 4.1 respectively. It was later found that one. Observation has been erroneously recorded as 40 instead of the correct value 30. Compute the correct mean and the standard deviation of the 100 observations.

10. The mean marks of schools $A$ and $B$ for test given to their students are 31 and 45 respectively. The standard deviation of the distribution of marks of school $A$ is 5. To compare the results the marks of school $B$ are scaled by using a linear transformation so that the mean and the standard deviation of school $B$ are the same as those of school $A$ and a mark of 85 at school $B$ becomes 63 under the transformation. Find the linear transformation and hence. Find the original standard deviation of the distribution of marks of school $B$.

## Part B

❖ Answer Five questions only.

11. (a) A particle $P$ is projected at a point $O$ vertically upward under the gravity with velocity $u$. After time $\frac{u}{2g}$, another particle $Q$ is projected at the point $O$ vertically upward under the gravity with velocity $v(> u)$. Let $A$ be the highest point that the particle $P$ reaches. The particles $P$ and $Q$ meet at the point $A$. Draw the velocity-time graphs for the complete motions of the particles $P$ and $Q$ in the same figure.

Using these velocity-time graphs show that

(i) $OA = \frac{u^2}{2g}$,

(ii) $v = \frac{5u}{4}$ and the velocity of the particle $Q$ at the point $A$ is $\frac{3u}{4}$.

(iii)when the particle $Q$ reaches the highest point the height of the particle $P$ from the point $O$, is $\frac{7u^2}{32g}$.

(b) A car of mass $M\ kg$ is travelling on a level road against a resistant $R$ to the motion which is a constant at all speeds. If the maximum power of the engine is $H\ kW$ and the car has a maximum speed of $v\ ms^{-1}$ on a level road, find the resistance $R$ in terms of $M, H$ and $v$.

Find the acceleration of the car in terms of $M, Hv, g$ and $\alpha$ when it is moving.

(i) at speed $\frac{v}{3}\ ms^{-1}$ directly up,

(ii) at speed $\frac{v}{2}\ ms^{-1}$directly down

along a straight road inclined at an angle $\alpha$ to the horizontal.

If the acceleration of the car in case (ii) is twice that in case (i), find $\sin\alpha$ in terms of $M, H, v$ and $g$.

In this case find the maximum speed in terms of $v$ that can be made by the car when moving directly up the road.

12. (a) A particle is projected under the gravity in a vertical plane with a velocity $u$ at an angle $\theta$ to the horizontal, at a point $C$ which is at a height $k$ from a point $O$. Consider a rectangular Cartesian system of coordinates by taking horizontal and vertical lines through the point $O$ in the plane of projection as $Ox$ and $Oy$ axes respectively. If at time $t$ the particle is at the point $(x, y)$, show that $y = k + x\tan\theta - \frac{gx^2sec^2\theta}{2u^2}$.

A particle $P$ is projected under the gravity in the vertical plane at the point $A(0, h)$, where $h$ is positive, with a velocity $v$ at an angle $\alpha$ the horizontal. At the same instant another particle $Q$ is projected under the gravity in the vertical plane at the point $B\left(0, \frac{h}{2}\right)$ with a velocity $w$ at an angle $\beta(> \alpha)$ to the horizontal. If the two particles $P$ and $Q$ meet at a point whose horizontal distance is $d$, show that $v\cos\alpha = w\cos\beta$ and $h = 2d(\tan\beta - \tan\alpha)$.

Show also, that the time taken for the two particles to meet is $\frac{h}{2(w\sin\beta - v\sin\alpha)}$.

(b) One end of a light inextensible string is attached to a ceiling which is at a height of 3 metres from a horizontal floor. The string passes under a smooth light movable pulley $P$ to which a particle of mass $m$ is fixed and then over a smooth light pulley fixed to the ceiling. A particle $Q$ of mass $M(> m)$ is attached to the other end of the string. When the movable pulley $P$ and the particle $Q$ are heights $\frac{1}{2}$ metres and 1 metre respectively from the floor and the portions of the string not in contact with pulleys are vertical, the system is released from rest.

Find the acceleration of the particle $Q$ and the tension of the string

Show that the particle $Q$ will reach the floor after time $\sqrt{\frac{4M+m}{(2M-m)g}}$ seconds and the pulley $P$ will rise to a height $\frac{1}{2} + \frac{3M}{4M+m}$ metres from the floor.

13. $A$ and $B$ are two points on a smooth horizontal table at a distance $8l$ apart. A smooth particle $P$ of mass $m$ lies at a point on $AB$ in between the points $A$ and $B$. The particle $P$ is attached to the point $A$ by a light elastic string of natural length $3l$ and modulus of elasticity $4\lambda$ and to the point $B$ by a light elastic string of natural length $2l$ and modulus of elasticity $\lambda$.

If the particle $P$ is in equilibrium at a point $C$, show that $AC = \frac{42}{11}l$.

The particle $P$ is held at the mid-point $M$ of $AB$ and then is released from rest. When the particle $P$ is at a distance from the point $A$ along $AB$, obtain the tensions of the two strings.

Write down the equation of motion of the particle $P$ for $\frac{40}{11}l \leq x \leq 4l$ and show, in the usual notation that $x + \frac{11\lambda}{6ml}\left(\ddot{x} - \frac{42}{11}l\right) = 0$.

By writing $y = x - \frac{42}{11}l$ show that $\ddot{y} + \frac{11\lambda}{6ml}y = 0$.

Assuming that the solution of the above equation is of the form $y = A\cos\omega t + B\sin\omega t$, find the constant $A, B$ and $\omega$.

Find the velocity of the particle $P$ when it is at a point, distant $\frac{41}{11}l$ from the point $A$.

14. (a) Let $A$ and $B$ be two distinct not collinear with a point $O$. Let the position vectors of the points $A$ and $B$ with respect to the point $O$ be $a$ and $b$ respectively. If $D$ is the point on $AB$ such that $BD = 2DA$, show that the position vector of the point $D$ with respect to the point $O$ is $\frac{1}{3}(2a + b)$.

If $\overrightarrow{BC} = ka (k > 1)$ and the points $O, D$ and $C$ are collinear, find the value of $k$ and the ration $OD : DC$ .

Express $\overrightarrow{AC}$ in terms of $a$ and $b$.

Further, if the line through the point $O$ parallel to $AC$ meets $AB$ at $E$, show that $6DE = AB$.

(b) The coordinates of the points $A, B$ and $C$ with respect to a rectangular Cartesian axes $Ox$ and $Oy$, are $(\sqrt{3}, 0), (0, -1)$ and $\left(\frac{2\sqrt{3}}{3}, 1\right)$ respectively. Forces of magnitude $6P, 4P, 2P$ and $2\sqrt{3P}$ newtons act along $OA, BC, CA$ and $BO$ respectively in the directions indicated by the order of the letters. Find the magnitude and the direction of the resultant of these forces.

Find the point at which the line of action of the resultant cuts the $y$-axis.

Hence, find the equation of the line of action of the resultant.

Another force of magnitude $6\sqrt{3}P$ newtons is introduced to the system along $AB$ in the direction indicated by the order of the letters. Show that the system is reduced to a couple of magnitude $10P$ newton metre.

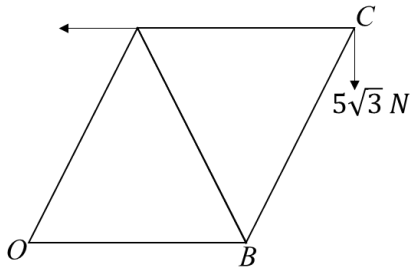
15. (a) Two equal uniform rods $AB$ and $AC$ each of weight $W$ are freely jointed at $A$ and the ends $B$ and $C$ are connected by a light inextensible string. The rods are kept in equilibrium in a vertical plane with the ends $B$ and $C$ are on two smooth planes each of which inclined at an angle $\alpha$ to the horizontal; $BC$ being horizontal and $A$ being above $BC$. Find the reaction at $B$.

If $\tan\theta > 2\tan\alpha$, where $B\hat{A}C = 2\theta$, then show that the tension of the string is $\frac{1}{2}W(\tan\theta - 2\tan\alpha)$.

Find the reaction at the joint $A$.

(b) Five light equal rods $OA, OB, AC, AB$ and $BC$ are smoothly jointed at their ends to form a framework as shown in the figure.

The framework is smoothly hinged at $O$ and carries a weight of $5\sqrt{3}$ newtons at $C$. The framework is held in a vertical plane, with $OB$ horizontal by a horizontal force of $P$ newtons at $A$.

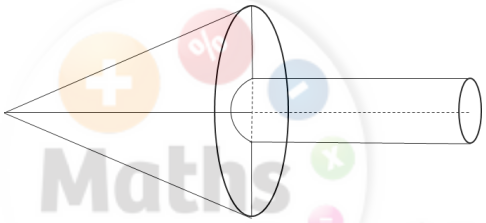



(i) Find the value of $P$.

(ii) Find the magnitude and the direction of the reaction at $O$.

(iii) Using Bow's notation, draw a stress diagram for the framework and find the stresses in all rods, distinguishing between tensions and thrusts.

16. Show that the centre of mass of a uniform solid right circular cone of height $h$ is on its axis of symmetry at a distance $\frac{1}{4}h$ from the base of the cone.



A uniform solid composite body consists of a right circular cone of base radius $3r$ and height $h$ and a right circular cylinder of radius $r$ and height $2h$ fixed together as shown in the figure.

Show that the centre of mass of the composite body is on its axis of symmetry at a distance $\frac{5}{4}h$ from the vertex of the cone.

The composite body is hanged freely in a vertical plane by a light inextensible string, one end of which is fixed to a ceiling and the other end to a point $A$ on the circumference of the circular base of the cone.

If the axis of symmetry of the composite body makes an angle $\alpha$ with the downward vertical, show that, $\tan\alpha = \frac{12r}{h}$.

By applying along the axis of symmetry of the composite body, a force $P$ at the vertex of the cone, the composite body is kept in equilibrium so that the axis of symmetry of the composite body is horizontal. Find the force $P$ and the tension of the string in terms of $W$ and $\alpha$, where $W$ is the weight of the composite body.

17. (a) An urn contains 5 white, 3 black and 7 red similar balls. Three balls are taken from the urn at random without replacement.

Find the probability that,

(i) all three balls are black,

(ii) none of the three balls is white,

(iii)at least one ball is white,

(iv)the balls are of different colours,

(v) the three balls are taken in the order black, red then white.

(b) Students in a certain class were given a question paper in statistics. The marks obtained by these students are given in the following grouped frequency table:

| Range of Marks | Number of Student |
|---|---|
| 00-20 | 14 |
| 20-40 | $f_1$ |
| 40-60 | 27 |
| 60-80 | $f_2$ |
| 80-100 | 15 |

The frequencies of the marks ranges $20-40$ and $60-80$ are missing in the table. However, the mode and the median of the grouped frequency are known as 48 and 50 respectively.

Calculate the two missing frequencies in the table.

Hence, obtain the total number of students who sat for the statistics paper.

Find the mean and the standard deviation of the grouped frequency distribution.