Combined Mathematics I – 2011 Advanced Level Examination

Part A

01. Using the Principle of Mathematical Induction, prove that $n^3 + 5n$ is divisible by 3 for every $n \in \mathbb{Z}^+$

02. Find how many numbers between 2000 and 4000 can be formed using the digits $1, 2, 3$ and 4, if repetitions of the digits are (i) not allowed, (ii) allowed.

03. Using the binomial expansion for a positive integral index, show that $(1+\sqrt{3})^6 + (1-\sqrt{3})^6 = 416$. Hence, find the integer part of $(1+\sqrt{3})^6$.

04. Show that $\lim_{x\to 0} \frac{\sqrt{4+3\sin x}-\sqrt{4-3\sin x}}{2x} = \frac{3}{4}$.

05. Find constants A and B such that $\frac{d}{dx}\{e^{2x}(A\sin 3x + B\sin 3x)\} = 13e^{2x}\sin 3x$.
Hence, find $\int e^{2x}\sin 3x dx$.

06. Find the equation of the straight line parallel to the straight line $3y + 2x + 5 = 0$ and passing through the point that divides the straight line joining the points $(2, 3)$ and $(-1, 2)$ externally in the ratio $3:2$.

07. A curve is given by $x = 3t, y = \frac{3}{t}$, where t is a non-zero parameter. Show that the equation of the tangent to the curve at the point $\left(3t, \frac{3}{t}\right)$ is $x + t^2 y = 6t$.
Deduce that, as t varies, the area of the triangular region bounded by the coordinate axes and this tangent is a constant.

08. Find the equations of the two circles, each of radius $\sqrt{2}$, touching the straight line $x + y + 1 = 0$ and having the centres on the y-axis.

09. The length of the tangent from a point P to the circle $x^2 + y^2 - 12x = 0$ is twice the length of the tangent from the point P to the circle $x^2 + y^2 - 9 = 0$. Show that the point P lies on the circle $x^2 + y^2 + 4x - 12 = 0$.

10. The sides of a triangle are $P-1, P$ and $P+1$, where P is a real number such that $P>1$. If the largest angle of the triangle is twice the smallest angle of the triangle, using the Sine rule and the cosine rule find the value of P.

Part B

❖ Answer five questions only.

11. (a) Let α and β be the roots of the quadratic equation $ax^2+bx+c=0$ where a, b and c are real numbers.

Show that α and β are both

(i) real, if and only if $b^2-4ac \geq 0$.

(ii) purely imaginary, if and only if $b=0$ and $ac>0$.

Find the quadratic equation whose roots are α^2 and β^2.

Show that the roots of this quadratic equation are both real, if and only if either α and β are both real or α and β are both purely imaginary.

(b) Let $f(x)=x^3-3abx-(a^3+b^3)$, where a and b are real numbers. Show that $(x-a-b)$ is a factor of $f(x)$. Find the other factor of $f(x)$ in quadratic form.

Hence or otherwise, show that if a and b are distinct, then $f(x)=0$ has only one real root.

Deduce that $x^3-9x-12=0$ has only one real root and find it.

12. (a) Let $U_r=\frac{1}{(2r-1)(2r+1)(2r+3)}$ for $r \in \mathbb{Z}^+$.

Find $\frac{U_{r+1}}{U_r}$ terms of r.

Hence, show that $(2r-1)U_r-(2r+1)U_{r+1}=4U_{r+1}$ for $r=1,2,3,\ldots..$

Deduce that $\sum_{r=1}^{n} U_r=\frac{1}{12}-\frac{1}{4(2n+1)(2n+3)}$

Is the series $\sum_{r=1}^{\infty} U_r$ convergent? Justify your answer.

(b) Draw the graph of $y=|2x-8|$.

Hence, draw the graph of $y=-|2x-8|$.

Draw the graphs of $y=4-|2x-8|$ and $y=|2x-10|$ in the same figure.

Hence or otherwise, find the set of real values of x satisfying the inequality $|2x-10|+|2x-8| \leq 4$.

13. (a) Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $\lambda, \mu \in \mathbb{R}$. Find the values of λ and μ such that $A(\lambda A + \mu I) = I$, where I is the 2×2 identity matrix.

Hence, find A^{-1}.

(b) Let P, Q and R be three dictinct point which represent complex numbers z_0, z_1 and z_2 respectively in the Argand diagram.

If $PQ = PR$ and θ is the angle measured from PQ to PR in the anti-clockwise sense, show that $z_2 - z_0 = (z_1 - z_0)(\cos\theta + i\sin\theta)$.

The points A, B, C and D taken in the anti-clockwise sense, form a square in the Argand diagram. Let $1 - i$ and z be the complex numbers represented by the points A and B repectively. Find the complex numbers represented by the points C and D in terms of z.

If C varies such that $AC = 2$, find the locus of B in the Argand diagram.

14. (a) Let $f(x) = 2x^3 + ax^2 + bx$ for $x \in \mathbb{R}$, where a and b are real constants.

Suppose that $f'(3) = 12$ and $f''(3) = 18$, where f' and f'' have the usual meaning.

Find the values of a and b.

For these values of a and b, sketch the graph of $y = f(x)$ indicating the turning point.

Hence, find the number of solutions of the equation $2x^2 + ax + b = \frac{3}{x}$.

(b) A closed rectangular box with a square base is made of thin cardboard. The volume of the box is $8192\ cm^3$. Let the length of a side of the square base be $4x cm$. A circular hole of radius is $x\ cm$ cut out from the top square face. Show that the surface area $A\ cm^2$ of the box with the hole is given by $A = (32 - \pi)x^2 + \frac{8192}{x}$.

Hence, show that A is minimum when $x = \frac{16}{\sqrt[3]{32-\pi}}$.

15. (a) Using the method of Integration by parts, evaluate $\int_1^e x^{\frac{3}{2}} \ln x \, dx$

(b) Let $t = \tan x$

Show that $\cos 2x = \frac{1-t^2}{1+t^2}$, $\sin 2x = \frac{2t}{1+t^2}$ and $\frac{dx}{dt} = \frac{1}{1+t^2}$.

Hence, show that $\int_1^{\frac{\pi}{4}} \frac{1}{4\cos 2x + 3\sin 2x + 5} dx = \frac{1}{12}$

(c) Let a and b be distinct real numbers. Find constants A and B such that

$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ for $x \in \mathbb{R} - \{a, b\}$

By replacing x, a and b appropriately in the above equation, write down $\frac{1}{(x^2+a^2)(x^2+b^2)}$ in partial fractions and hence, find $\int \frac{1}{(x^2+a^2)(x^2+b^2)}dx$

16. (a) Show that the equations of the two straight lines drawn through the origin perpendicular to each other so as to form an isosceles right angled triangle with the straight line $lx + my + 1 = 0$ are $(l-m)x + (l+m)y = 0$ and $(l+m)x - (l-m)y = 0$.

(b) Show that, if the circle

$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ cut the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at the ends of a diameter of the circle $S = 0$, then $2g^2 + 2f^2 - c = 2gg' + 2ff' - c'$.

A variable circle cuts each of the circles $S_1 \equiv x^2 + y^2 - 25 = 0$ and $S_2 \equiv x^2 + y^2 - 2x - 4y - 11 = 0$ at the ends of a diameter. Show that the centre of the variable circle lies on the straight line $x + 2y + 2 = 0$.

17. (a) Using the identity $cos^2\theta + sin^2\theta = 1$ or otherwise, determine the real constants a and b such that $cos^6\theta + sin^6\theta = a + b\cos 4\theta$.

Hence or otherwise,

(i) Sketch the graph of $y = 8(cos^6x + sin^6x)$,

(ii) find the general solution of the equation $cos^6x + sin^6x = \frac{5}{4} + \frac{1}{2}\sin 4x$.

(b) Solve the equation $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$.