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G.C.E. (Advanced Level) Examination - August 2011

Combined Mathematics - I

Scheme of Marking

Part A

(01) Let $P(n) = n^3 + 5n$

When n = 1, P(1) = 1 + 5 = 6 Which is divisible by 3

Thus, the result is true for n = 1 (5)

Assume that the result is true for n = k

i.e. assume that $P(k) = k^3 + 5k$ is divisible by 3 (5)

Let n = k + 1

Then $P(k+1) = (k+1)^3 + 5(k+1)$

$= K^3 + 3k^2 + 3k + 1 + 5k + 5$ (5)

$= K^3 + 5k + 3(k^2 + 3k + 2)$

is divisible by 3 since $K^3 + 5k$ is divisible by 3.

Thus, the result is true for n = k + 1 (5)

By the principle of Mathematical Induction $n^3 + 5n$ is divisible by 3 for every $n \in z^+$ (5) **25**

(02) (i)

There are 2 ways of finding the first digit and 3 x 2 x 1 = 6 ways of selecting the other three digits. (5)

Thus, there are 2 x 6 = 12 ways of selecting a number between 2000 and 4000 using the digits 1, 2, 3 and 4 without the repetition of the digits. (5)

(ii) There are 2 ways of finding the first digit and 4x4x4 = 64 ways of selecting the other three digits. (10)

Thus there are 2 x 64 = 128 ways of selecting a number between 2000 and 4000 using the digits 1, 2, 3 and 4, with the repetition of the digits. (5) **25**

(03)

$$(1+\sqrt{3})^6 = \begin{pmatrix} {}^6C_0 + {}^6C_1\sqrt{3} + {}^6C_2(\sqrt{3})^2 + {}^6C_3(\sqrt{3})^3 + \\ {}^6C_4(\sqrt{3})^4 + {}^6C_5(\sqrt{3})^5 + {}^6C_6(\sqrt{3})^6 \end{pmatrix} (5)$$

$$(1-\sqrt{3})^6 = \begin{pmatrix} {}^6C_0 - {}^6C_1\sqrt{3} + {}^6C_2(\sqrt{3})^2 - {}^6C_3(\sqrt{3})^3 \\ + {}^6C_4(\sqrt{3})^4 - {}^6C_5(\sqrt{3})^5 + {}^6C_6(\sqrt{3})^6 \end{pmatrix} (5)$$

$$(1+\sqrt{3})^6 - (1-\sqrt{3})^6 = 2\begin{Bmatrix} {}^6C_0 + {}^6C_2(\sqrt{3})^2 \\ + {}^6C_4(\sqrt{3})^4 + {}^6C_6(\sqrt{3})^6 \end{Bmatrix}$$

$= 2\{1 + 15\times3 + 15\times9 + 27\}$ (5)

$= 2\{1 + 180 + 27\} = 416$ (5)

$(1+\sqrt{3})^6 = 416 - (1-\sqrt{3})^6$

$0 < (1-\sqrt{3})^6 < 1 \quad \therefore 0 < \sqrt{3} - 1 < 1$

Thus, the integer part of $(1+\sqrt{3})^6$ is 415. (5) **25**

(04)

$$= \lim_{x\to0} \frac{\left[\sqrt{4+3\sin x} - \sqrt{4-3\sin x}\right]}{2x} \times \frac{\left[\sqrt{4+3\sin x} + \sqrt{4-3\sin x}\right]}{\sqrt{4+3\sin x} + \sqrt{4-3\sin x}}$$ (5)

$$= \lim_{x\to0} \frac{(4+3\sin x) - (4-3\sin x)}{2x} \frac{1}{\left[\sqrt{4+3\sin x} + \sqrt{4-3\sin x}\right]}$$ (5)

$$= 3\lim_{x\to0} \frac{\sin x}{x} \times \lim_{x\to0} \frac{1}{\sqrt{4+3\sin x} + \sqrt{4-3\sin x}}$$ (5)

$$= 3\times1\times\left[\frac{1}{2\sqrt{4}}\right] = \frac{3}{4}$$ (5) **25**

(05)

$$\frac{d}{dx}\left\{e^{2x}(A\sin3x + B\cos3x)\right\}$$

$$= e^{2x}(3A\cos3x - 3B\sin3x) + 2e^{2x}(A\sin3x + B\cos3x)$$ (5)

$$= e^{2x}\{(3A+2B)\cos3x + (2A-3B)\sin3x\}$$

$= 13e^{2x}\sin3x$ if and only if $3A + 2B = 0$ and $2A - 3B = 13$

$2A - 3B = 13 \Leftrightarrow A = 2$ and $B = -3$ (5)

Hence $\int e^{2x}\sin3x\,dx = e^{2x}\left[\frac{2}{13}\sin3x - \frac{3}{13}\cos3x\right] + C$ (5) (5)

where C is an arbitary constant. **25**

(06) Since the required straight line Parallel to the straight line $3y + 2x + 5 = 0$, the equation of the required straigh line can be written as $3y + 2x + c = 0$, where the constant C to be determined Let (x_0, y_0) be the co-ordinates of the point that divides the straight line joining the point (2, 3) and (-1, 2) externally in the ratio 3 : 2. (5)

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Then $x_0 = \frac{2\times 2+(-3)\times(-1)}{-3+2} = -7$ and (5)

$y_0 = \frac{2\times 3+(-3)\times(+2)}{-3+2} = 0$ (5)

Since the point (-7, 0) lies on the line we have
$3(0) + 2(-7) + C = 0$ Which implies $C = 14$ (5)
Thus, the equation of the required straight line is
$3y + 2x + 14 = 0$ (5) **25**

(07) $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{-3}{t^2}\cdot\frac{1}{3} = \frac{1}{t^2}$ (5)

The equation of the tangent is $\frac{\left(y-\frac{3}{t}\right)}{x-3t} = -\frac{1}{t^2}$ (5)

i.e. $x + t^2y = 6t$ (5)

This tangent cuts the x - axis at the point $A = \left(0, \frac{6}{t}\right)$

and the y - axis at the point B = 6t, o (5)

The area of the triangle OAB is $\frac{1}{2}\times 6t\times\left(\frac{6}{t}\right) = 18sq$ (5)

Unit which is independent of t. **25**

(08) Centre of each circle is of the form (o, b) (5)
The perpendicular distance from (o, b)

to line $x + y + 1 = 0$ is $\left|\frac{b+1}{\sqrt{2}}\right| = \sqrt{2}$ (5)

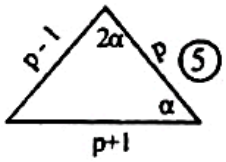
$\Rightarrow (b+1)^2 = 4 \Rightarrow b^2 + 2b - 3 = 0 \Rightarrow b = 1$ and -3
Thus, the equation of the two circles are (5)
$x^2 + (y-1)^2 = 2$ *and* $x^2 + (y+3)^2 = 2$
(5) (5) **25**

(09) Let $P = (\alpha, \beta)$
The length of the tangent from a point P to the circle
$x^2 - 12x + y^2 = 0$ is $\sqrt{\alpha^2 - 12\alpha + \beta^2}$ (5)

The length of the tangent from a point P to the circle
$x^2 + y^2 - 9 = 0$ is $\sqrt{\alpha^2 + \beta^2 - 9}$ (5)

$\alpha^2 - 12\alpha + \beta^2 = 4(\alpha^2 + \beta^2 - 9)$ (5)
i.e. $3\alpha^2 + 3\beta^2 + 12\alpha - 36 = 0$
i.e. $\alpha^2 + \beta^2 + 4\alpha - 12 = 0$ (5)
Thus the point P, lies on the circle
$x^2 + y^2 + 4x - 12 = 0$ (5) **25**

(10) Using sine rule We have



$\frac{\sin\alpha}{p-1} = \frac{\sin 2\alpha}{p+1}$ (5)

$\Rightarrow \cos\alpha = \frac{p+1}{2(p-1)}$ (5)

Using cosine rule we have

$\cos\alpha = \frac{(p+1)^2 + p^2 - (p-1)^2}{2p(p+1)} = \frac{p+4}{2(p+1)}$ (5)

Thus, we have

$\frac{p+1}{2(p-1)} = \frac{p+4}{2(p+1)} \Rightarrow p^2 + 2p + 1 = p^2 + 3p - 4$

$\Rightarrow P = 5$ (5) **25**

Part - B

(11) (a) (i)

$$ax^2 + bx + c = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$
$$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right\}$$
$$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right\}$$

$ax^2 + bx + c = 0$ if and only if

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \rightarrow (01)$ (5)

x has two real values satisfying equation (01) if and only if
$b^2 - 4ac \geq 0$ (5)

i.e. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ both real

if and only if $b^2 - 4ac \geq 0$ (5)

i.e. $ax^2 + bx + c = 0$ has real roots if and only if
$b^2 - 4ac \geq 0$ (5) **20**

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(ii) x has two complex values satisfying equation (01) if and only if $b^2 - 4ac < 0$ (5)

i.e. $x = \frac{-b \pm i\sqrt{b^2 - 4ac}}{2a}$ (both complex) if and only if

$b^2 - 4ac < 0$

$x = \frac{\sqrt{b^2 - 4ac}}{2a} i$ (both pure imaginary) if and only if

$b^2 - 4ac < 0$ and $b = 0$ (5)

i.e. $x = \frac{\sqrt{b^2 - 4ac}}{2a} i$ (both pure imaginary) if and only if

$ac > 0$ and $b = 0$ (5)

i.e. $ax^2 + bx + c = 0$ has pure imaginary roots if and only if $ac > 0$ and $b = 0$ (5) 20

Aliter for (a) (i)

Since α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$ we have

$\alpha + \beta = -\frac{b}{a} \rightarrow (01)$ and $\alpha\beta = \frac{c}{a}$ (5)

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \left(\frac{-b}{c}\right)^2 - 4\left(\frac{c}{a}\right) = \frac{b^2 - 4ac}{a^2}$ (5)

$\alpha - \beta = \pm\frac{\sqrt{b^2 - 4ac}}{a} \rightarrow (02)$ if and only if $b^2 - 4ac \geq 0$ (5)

(01) and (02) imply that the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$, and both are real. (5)

Both roots are real if and only if $b^2 - 4ac \geq 0$. 20

Aliter for (a) (ii)

$\alpha - \beta = \pm\frac{\sqrt{b^2 - 4ac}}{a} i \rightarrow (03)$ if and only if $b^2 - 4ac < 0$

(01) and (03) imply that the two roots of the quadratic equation $ax^2 + bx + c = 0$ are (5)

$\frac{-b + i\sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - i\sqrt{b^2 - 4ac}}{2a}$, and they are complex. (5)

If $b = 0$, the real part of each of them are zero and hence both of them are pure imaginary. (5)

i.e. both roots are pure imaginary if and only if $ac > 0$ and $b = 0$ (5) 20

The quadratic equation whose roots are α^2 and β^2, can be written as

$x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$ (5)

i.e. $x^2 - \left\{(\alpha + \beta)^2 - 2\alpha\beta\right\}x + \alpha^2\beta^2 = 0$

i.e. $x^2 - \left\{\left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right\}x + \left(\frac{c}{a}\right)^2 = 0$ (5)

i.e. $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$ (5) 15

The roots of this quadratic equation are both real if and only if $\left\{-\left(b^2 - 2ac\right)\right\}^2 - 4a^2c^2 \geq 0$ (5)

i.e. the roots are both real if and only if

$\left\{\left(b^2 - 2ac\right) - 2ac\right\}\left\{\left(b^2 - 2ac\right) + 2ac\right\} \geq 0$

i.e. the roots are both real if and only if

$b^2\left(b^2 - 4ac\right) \geq 0$ (5)

$b^2\left(b^2 - 4ac\right) \geq 0$ if and only if either $b^2 - 4ac \geq 0$ and b is any real value or $b^2 - 4ac < 0$ and $b = 0$ (10)

i.e. $b^2\left(b^2 - 4ac\right) \geq 0$ if and only if either α and β are both real or α and β are both purely imaginary. (5)

Hence, We have the result. 25

Aliter:

α and β and the roots of the quadratic equation $ax^2 + bx + c = 0$

Thus, α and β are both

(i) Real if and only if $b^2 - 4ac \geq 0$ (5)

(ii) Purely imaginary if and only if $ac > 0$ and $b = 0$ (5)

(iii) complex if and only if $b^2 - 4ac < 0$ and $b \neq 0$ (5)

α^2 and β^2 are both cannot be real if α and β are both complex (5)

Thus, α^2 and β^2 are both real if any only if α and β are both real or both purely imaginary. (5) 25

(b) $f(a + b) = (a + b)^3 - 3ab(a + b) - (a^3 + b^3) = 0$ (5)

Thus, $(x - a - b)$ is a factor of $f(x)$ 05

$f(x) = x^3 - 3abx - (a^3 + b^3) = (x - a - b)\left(x^2 + (a + b)x\right) + (a^2 - ab + b^2)$ (5) (5) (5)

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$f(x) = (x-a-b)\left\{x^2 + (a+b)x - 3ab + (a+b)^2\right\}$

If $f(x) = 0$ has only one real root then

$x^2 + (a+b)x - 3ab + (a+b)^2 = 0$ should have complex roots. (5)

For $x^2 + (a+b)x - 3ab + (a+b)^2 = 0$ to have complex roots we muust have

$(a+b)^2 - 4\left\{(a+b)^2 - 3ab\right\} < 0$ (5)

i.e. We must have $(a+b)^2 - 4ab > 0$

i.e. We must have $(a-b)^2 > 0$ (5)

i.e. a and b should be distinct (5) 20

If $ab = 3$ and $a^3 + b^3 = 12$ then

$x^3 - 9x - 12 = x^3 - 3abx - (a^3 + b^3)$

i.e. if $a^3 + \left(\frac{3}{a}\right)^3 = 12$ then $x^3 - 9x - 12 = x^3 - 3abx - (a^3 + b^3)$

i.e. if $a^6 - 12a^3 + 27 = 0$ then $x^3 - 9x - 12 = x^3 - 3abx - (a^3 + b^3)$

i.e. if $(a^3 - 9)(a^3 - 3) = 0$ Then $x^3 - 9x - 12 = x^3 - 3abx - (a^3 + b^3)$

i.e. if $(a^3 - 9)(a^3 - 3) = 0$ Then $x^3 - 9x - 12 = x^3 - 3abx - (a^3 + b^3)$

i.e. if $a = 3^{\frac{2}{3}}$ and $b = 3$ then (5) $x^3 - 9x - 12 = x^3 - 3abx - (a^3 + b^3)$

If $a = 3^{\frac{2}{3}}$ and $b = 3^{\frac{1}{3}}$ then $x^3 - 9x - 12 = 0$ can be written as $x^3 - 3abx - (a^3 + b^3)$ (5) (5)

Since $a = 3^{\frac{2}{3}}$ and $b = 3^{\frac{1}{3}}$ are distinct by above result (5)

$x^3 - 9x - 12 = 0$ has only one real root. 25

This real root is $3^{\frac{2}{3}} + 3^{\frac{1}{3}}$ (5) 05

(12) (a)

$u_r = \frac{1}{(2r-1)(2r+1)(2r+3)} \Rightarrow u_{r+1} = \frac{1}{(2r+1)(2r+3)(2r+5)}$

$\frac{u_{r+1}}{u_r} = \frac{(2r-1)(2r+1)(2r+3)}{(2r+1)(2r+3)(2r-5)} = \frac{(2r-1)}{(2r+5)}$ (5) 05

i.e. $(2r-1)u_r = (2r+5)u_{r+1}$

$= (2r+1)u_{r+1} + 4u_{r+1}$ (5)

i.e. $(2r-1)u_r - (2r+1)u_{r+1} = 4u_{r+1}$ for = 1, 2, 3..... 05

$4u_1 = 5u_1 - u_1$ (5)

When $r = 1$ $4u_2 = u_1 - 3u_2$ (5)

When $r = 2$ $4u_3 = 3u_2 - 5u_3$ (5)

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When $r = n-2$, $4u_{n-1} = (2n-5)u_{n-2} - (2n-3)u_{n-1}$ (5)

When $r = n-1$, $4v_n = (2n-3)u_{n-1} - (2n-1)u_n$ (5)

$4\sum_{r=1}^{n} u_r = 5u_1 - (2n-1)u_n = \frac{5}{1\cdot 3\cdot 5} - \frac{(2n-1)}{(2n-1)(2n+1)(2n+3)}$ (5) (5)

$\sum_{r=1}^{n} u_r = \frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$ (5) 40

yes the series $\sum_{r=1}^{\infty} u_r$ is convergent (5) 05

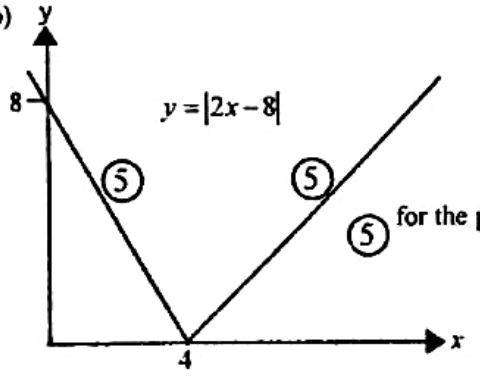
Reason :

$\lim_{n\to\infty} \sum_{r=1}^{n} u_r = \lim_{n\to\infty} \left\{\frac{1}{12} - \frac{1}{4(n-1)(2n+3)}\right\}$

$= \frac{1}{12} - \lim_{n\to\infty} \frac{1}{4n^2\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)} \Rightarrow \frac{1}{12}$ (5)

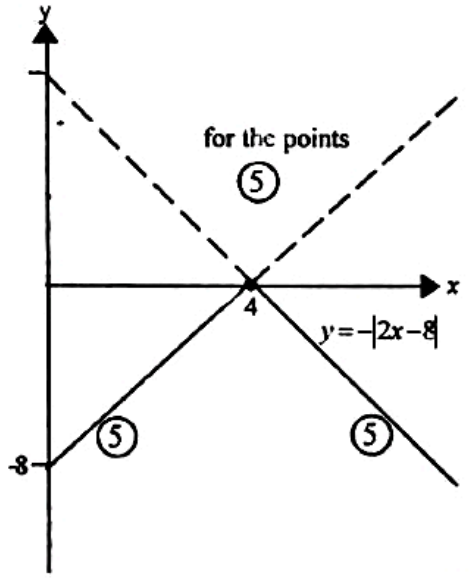
Since the sum of the first n terms of the series tends to a finite limit as n tends to infinity the series is convergent. (5) 15

(b)

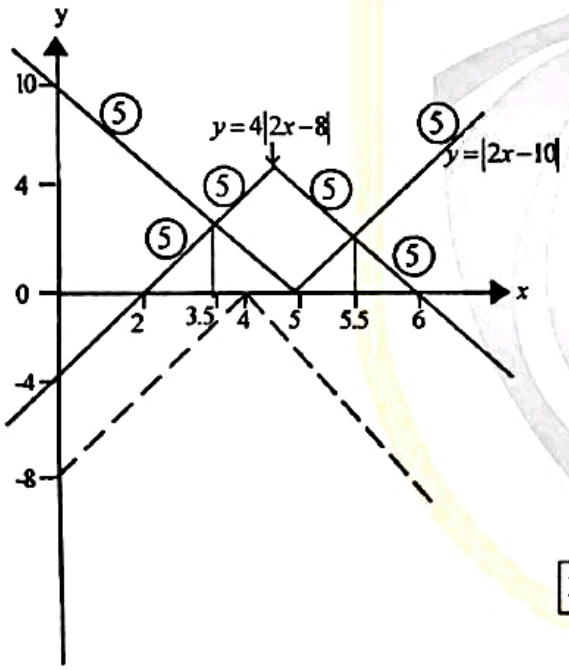


(5) (5) (5) for the points 15

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It is clear from the above graphs that the graph of $y = 4 - |2x - 8|$ is above the graph of $y = |2x - 10|$

When $3.5 \le x \le 5.5$ (10)

Thus $|2x+10|+|2x-8| \le 4$ hold When $3.5 \le x \le 5.5$ (10)

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(13) (a) $A(\lambda A + \mu I) = I \Rightarrow$

$$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\left\{\lambda\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}+\mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ (5)

i.e. $$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2\lambda+\mu & \lambda \\ -\lambda & 3\lambda+\mu \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ (10)

i.e. $$\begin{pmatrix} 3\lambda+2\mu & 5\lambda+\mu \\ -5\lambda-\mu & 8\lambda+3\mu \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$$ (10)

$3\lambda + 2\mu = 1$, (5) $5\lambda + \mu = 0$ (5) and $8\lambda + 3\mu = 1 \Rightarrow \lambda = -\frac{1}{7}$ (5) and $\mu = \frac{5}{7}$ (5)

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$$A^{-1} = A^{-1}A(\lambda A + \mu I) = \lambda A + \mu I$$ (5)

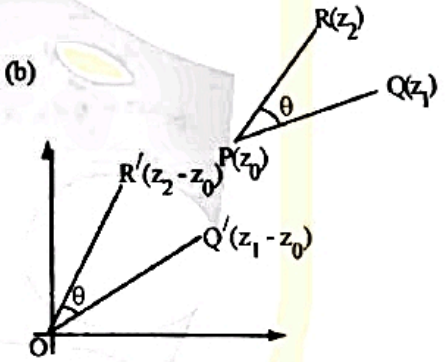
$$= \begin{pmatrix} 2\lambda+\mu & \lambda \\ -\lambda & 3\lambda+\mu \end{pmatrix}$$ (5)

$$= \begin{pmatrix} -\frac{2}{7}+\frac{5}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{3}{7}+\frac{5}{7} \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}$$ (5)

$$= \frac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$ (5)

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(b)



Draw the line segment OQ through O equal and parallel to PQ. (5)

Then the point Q^1 represents the complex number $z_1 - z_0$ in the Argand diagram (5) Similarly, draw the line segment OR^1 through O equal and parallel to PR. (5)

Then the pioint R^1 represents the complex number $z_1 - z_0$ in the Argand diagram. (5)

The point R^1 can obtained by rotating OQ^1 about O through an angle θ in the anti-clockwise sense. (5)

Thus the point R^1 represents the complex number $(z_1 - z_0)(\cos\theta + i\sin\theta)$, Hence,

We have $z_2 - z_0 = (z_1 - z_0)(\cos\theta + i\sin\theta)$ (5)

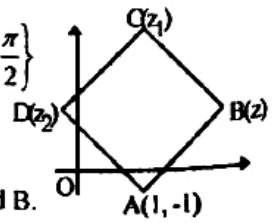
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Consider the three points A, B and D. Applying the above result We get

$$z_2 - (1-i) = \{z - (1-i)\}\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\}$$ (10)

i.e. $z_2 = i\{z - (1-i)\} + 1 - i$

i.e. $z_2 = iz - 2i = (z - 2)$ (5)



Consider the three points C, D and B.

Applying the above result again We get

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$z_1 - \{i(z-2) - z_1\}\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\}$ (10)

i.e. $z - z_1 = i\{i(z-2) - z_1\} = -(z-2) - iz_1$

i.e. $z_1 = \frac{2(z-1)}{1-i} = \frac{2(z-1)(1+i)}{(1-i)(1+i)} = (1+i)(z-1)$ (5)

$AC = |z_1 - (1-i)| = |(1+i)(z-1) - (1-i)| = |(1+i)z - 2|$ (5)

$= |(x-y-2) + i(x+y)|$ (5)

Since, $AC = 2$ We have $(x-y-2)^2 + (x+y)^2 = 4$ (5)

i.e. $x^2 + y^2 - 4x + 4y - 4 + x^2 + y^2 = 4$

i.e. $x^2 + y^2 - 2x + 2y = 0$

i.e. $(x-1)^2 + (y+1)^2 = 2$ (5) **50**

(14) (a) $f(x) = 2x^3 + ax^2 + bx$

$f'(x) = 6x^2 + 2ax + b$ and $f''(x) = 12x + 2a$ (5) (5)

Since $f'(3) = 12$ and $f''(3) = 18$ We then have

$54 + 6a + b = 12$ (5) and $36 + 2a = 18$ (5)

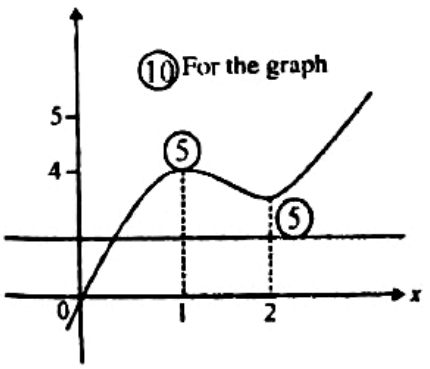
Thus We have $a = -9$ (5) and $b = 12$ (5) **30**

So we have $f(x) = 2x^3 - 9x^2 + 12x$ and

$f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2) = 0 \Leftrightarrow x = 1$ (5) and $x = 2$ (5)

	$x < 1$	$1 < x < 2$	$2 < x$
sign $f'(x)$	(+)	(-)	(+)
(10)	$f(x)$ is increasing	$f(x)$ is decreasing	$f(x)$ is increasing

Note that $f(0) = 0$, $f(1) = 5$ and $f(2) = 4$



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$2x^2 + ax + b = \frac{3}{x} \Leftrightarrow f(x) = 3$ (5)

The number of solutions of $2x^2 + ax + b = \frac{3}{x}$ is the same as the number of points of intersection of the curve $y = f(x)$ and the line $y = 3$ (5)

From the above graph we see that there is only one point of intersection and the required.

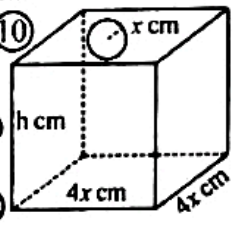
answer is 1. (5) **15**

(b) Let the height of the box be h cm.

$A = (2 \times 16x^2 + 4 \times 4xh) - \pi x^2$ (10)

The volume of the box is

$16x^2h$ and is equal to 8192. (10)



$16x^2h = 8192 \Rightarrow 16h = \frac{8192}{x^2}$ (5)

$A = (32-\pi)x^2 + \frac{8192}{x}$ (5) **30**

$\frac{dA}{dx} = 2(32-\pi)x - \frac{8192}{x^2}$ (5)

$\frac{dA}{dx} = 0 \Leftrightarrow x^3 = \frac{4096}{32-\pi} = \frac{(16)^3}{(32-\pi)} \Leftrightarrow x = \frac{16}{3\sqrt{32-x}}$ (5) (5)

$\frac{dA}{dx} < 0$, When, $x < \frac{16}{\sqrt{32-\pi}}$ and $\frac{dA}{dx} > 0$, When $x > \frac{16}{3\sqrt{32-\pi}}$ (5)

Thus, A is minimum, When $x = \frac{16}{3\sqrt{32-\pi}}$ (5) **30**

(15) (a)

$\int_1^e x^{\frac{3}{2}} \ell n\, x\, dx = \int_1^e \ell n\, x\, d\left(\frac{2}{5}x^{\frac{5}{2}}\right)dx$ (10)

$= \left[\frac{2}{5}x^{\frac{5}{2}}\ell nx\right]_1^e - \frac{2}{5}\int_1^e x^{\frac{5}{2}}\left(\frac{1}{x}\right)$ (10)

$= \frac{2}{5}e^{\frac{5}{2}} - \frac{2}{5}\int_1^e x^{\frac{3}{2}}dx \quad \therefore \ell ne = 1$ and $\ell nl = 0$ (5)

$= \frac{2}{5}e^{\frac{5}{2}} - \left(\frac{2}{5}\right)^2\left[x^{\frac{5}{2}}\right]_1^e$ (5)

$= \frac{2}{5}e^{\frac{5}{2}} - \left(\frac{2}{5}\right)^2 e^{\frac{5}{2}} + \left(\frac{2}{5}\right)^2$ (5) **35**

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(b)

$$\cos 2x - \cos^2 x - \sin^2 x = \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} = \frac{1-\tan^2 x}{1+\tan^2 x}$$

$$= \frac{1-t^2}{1+t^2} \quad (5)$$

$$\sin 2x = 2\cos x \sin x = \frac{2\cos x \sin x}{\cos^2 x + \sin^2 x}$$

$$= \frac{2\tan x}{1+\tan^2 x} = \frac{2t}{1+t^2} \quad (5)$$

$$t = \tan x \Rightarrow \frac{dt}{dx} = \sec^2 x = 1+\tan^2 x = 1+t^2$$

Thus, We have $\frac{dx}{dt} = \frac{1}{1+t^2}$ (5) **15**

$$\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2x + 3\sin 2x + 5}\,dx = \int_0^1 \frac{1}{4\left(1-t^2\right)+6t+5}\,dt \quad (15)$$

$$= \int_0^1 \frac{1}{t^2+6t+9}\,dt$$

$$= \int_0^1 \frac{1}{(t+3)^2}\,dt \quad (5)$$

$$= \left[-\frac{1}{(t+3)}\right]_0^1 \quad (5)$$

$$= -\frac{1}{4} + \frac{1}{3}$$

$$= \frac{1}{12} \quad (5)$$ **30**

(c) Let $A = \frac{1}{a-b}$ and $B = \frac{1}{b-a}$ (5) (5)

Then $\frac{A}{x-a} + \frac{B}{x-b} = \frac{1}{a-b}\left[\frac{1}{x-a} - \frac{1}{x-b}\right] = \frac{1}{(x-a)(x-b)}$ (5)

for $x \in \mathbb{R} - \{a,b\}$

Thus

$$\frac{1}{(x-a)(x-b)} = \frac{1}{(a-b)} \times \frac{1}{(x-a)} - \frac{1}{(a-b)(x-b)} \rightarrow (1) \quad (5)$$

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Replacing x by x^2 a by $(-a^2)$b by$(-b^2)$ in We get

$$\frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)} = \frac{1}{\left(-a^2+b^2\right)\left(x^2+a^2\right)} - \frac{1}{\left(-a^2+b^2\right)\left(x^2+b^2\right)} \quad (15)$$

$$\therefore \int \frac{1}{\left(x^2+a^2\right)\left(x^2-b^2\right)}\,dx$$

$$= -\frac{1}{\left(b^2-a^2\right)}\int \frac{1}{\left(x^2+a^2\right)}\,dx + \frac{1}{\left(b^2-a^2\right)}\int \frac{1}{x^2+b^2}\,dx \quad (5)$$

$$= \frac{1}{\left(b^2-a^2\right)}\left\{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) - \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right\} + C$$

If $a, b \neq 0$

Where C is an arbitary constant. (20)

If a = 0 Then $b \neq 0$

$$\therefore \int \frac{1}{x^2\left(x^2+b^2\right)}\,dx = \frac{1}{b^2}\int \frac{1}{x^2}\,dx - \frac{1}{b^2}\int \frac{1}{x^2+b^2}\,dx$$

$$= -\frac{1}{b^2 x} - \frac{1}{b^3}\tan^{-1}\left(\frac{x}{b}\right) + C'$$

Where C' is an arbitary constant.

If b = 0 Then $a \neq 0$

$$\therefore \int \frac{1}{x^2\left(x^2+b^2\right)}\,dx = -\frac{1}{a^2}\int \frac{1}{\left(x^2+a^2\right)}\,dx + \frac{1}{a^2}\int \frac{1}{x^2}\,dx \quad (5)$$

$$= -\frac{1}{a^2 x} - \frac{1}{a^3}\tan^{-1}\left(\frac{x}{a}\right) + C''$$

Where C" is an arbitary constant. (5)

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(16) (a) The equation of a line through the origin can be written as y = kx (5)

If this line intersects the line $\ell x + my + 1 = 0$ then the x coordinate of the point of intersection satisfies the equation $\ell x + m(kx) + 1 = 0$ which implies $x = -\frac{1}{\ell + mk}$ (10)

The coordinates of the point of intersection of the line y = kx and the line $\ell x + my + 1 = 0$ are

$$\left[-\frac{1}{\ell+mk}, -\frac{k}{\ell+mk}\right] \quad (5)$$

The distance from the origin to this Point of intersection

$$= \sqrt{\left(-\frac{1}{\ell+mk}\right)^2 + \left(-\frac{k}{\ell+mk}\right)^2} = \frac{\sqrt{1+k^2}}{|\ell+mk|} \quad (5) \quad (5)$$

Replacing k by $-\frac{1}{k}$ in $\frac{\sqrt{1+k^2}}{|\ell+mk|}$ (5) the distance from the origin to the point of intersection of the perpendicular to the y = kx and the line $\ell x + my + 1 = 0$ is obtained as

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$\frac{\sqrt{1+\left(-\frac{1}{k}\right)^2}}{\left|\ell+m\left(-\frac{1}{k}\right)\right|}$ (5) i.e. $\frac{\sqrt{1+k^2}}{|k\ell-m|}$ (5)

Since the triangle formed is isosceles we have

$\frac{\sqrt{1+k^2}}{|\ell+mk|}=\frac{\sqrt{1+k^2}}{|k\ell-m|}$ (5)

Thus, We have $k=+\frac{\ell+m}{\ell-m}$ (5)

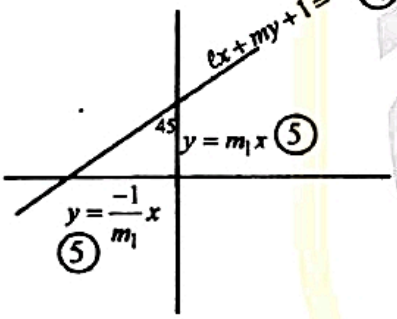
$\therefore -\frac{1}{k}=-\left(\frac{\ell-m}{\ell+m}\right)$ (5)

The equation of the two straight lines are

$(\ell-m)x+(\ell+m)y=0$ and $(\ell+m)x-(\ell-m)y=0$
(5) (5)

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(10)

$\tan 45=\left|\frac{m_1-\left(-\ell/m\right)}{1+m_1\left(-\ell/m\right)}\right|$ (5)
(5)

$1=\left|\frac{m_1+\frac{\ell}{m}}{1-\frac{m_1\ell}{m}}\right|$

$\mp=\frac{mm_1+\ell}{m-m_1\ell}$ (5)

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$1=\frac{mm_1+\ell}{m-m_1\ell}$

$m-m_1\ell=mm_1+\ell$

$\frac{m-\ell}{\ell+m}=m_1$ (5)

$\therefore y=m_1x$ හි සමීකරණය

$y=\left(\frac{m-\ell}{m+\ell}\right)x$

$(\ell+m)y=(m-\ell)x$

$(\ell+m)y+(\ell-m)x=0$ (5)

+ සැලකීමෙන්

$\frac{mm_1+\ell}{m-m_1\ell}=-1$ (5)

$mm_1+\ell=-m+m_1\ell$

$m_1(m-\ell)=-m-\ell$

$m_1=\frac{\ell+m}{\ell-m}$ (5)

$y=m_1x$ හි සමීකරණය

$y=\frac{(\ell+m)}{(\ell-m)}x$

$(\ell+m)x-(\ell-m)y=0$ (5)

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(b) The equation of the commen chord of the two circles $S=0$ and $S'=0$ is

$2(g-g^1)x+2(f-f^1)y+c-c^1=0$ (5)

Since the circle $S'=0$ cuts the circle $S=0$ at the ends of a diameter of the circle $S=0$ the centre $(-g,-f)$ of the circle $S=0$ should be on the line.

$2(g-g^1)x+2(f-f^1)y+c-c^1=0$ (5)

Thus, We have $2(g-g^1)+2(f-f^1)(-f)+c-c^1=0$ (5)

i.e. $2g^2+2f^2-c=2gg^1+2ff^1-c^1$ (5)

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Let $S\equiv x^2+y^2+2gx+2fy+c=0$ be the equation of the variable circle. (5)

Since the circle $S=0$ cuts the circle $S_1=0$ at the ends of a diameter of the circle S_1-0 We have

$-(-25)=-C\Rightarrow C=-25$
(15) (5)

Since the circle $S=0$ cuts the circle $S_2=0$ at the ends of a diameter of the circle $S_2=0$ We have

$2(-1)^2+2(-2)^2-(-11)=-2g+2(-2)f-C$ (15)

i.e. $2g+4f=-C-21$ (5)

Substituting for C We get $g+2f=2$ (5)

$-g+2(-f)=-2$ (5)

Thus, The centre $(-g,-f)$ of the variable circle lies on the line $x+2y+2=0$ (5)

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(17) (a) $\cos^2\theta + \sin^2\theta = 1 \Rightarrow (\cos^2\theta + \sin^2\theta)^3 = 1$ (5)

i.e. $\cos^6\theta + 3\cos^4\theta \sin^2\theta + 3\cos^2\theta \sin^4\theta + \sin^6\theta = 1$ (10)

i.e. $\cos^6\theta + 3\cos^2\theta \sin^2\theta (\cos^2\theta + \sin^2\theta) + \sin^6\theta = 1$

i.e. $\cos^6\theta + 3\cos^2\theta \sin^2\theta (\cos^2\theta + \sin^2\theta) + \sin^6\theta = 1$

i.e. $\cos^6\theta + \frac{3}{4}(2\cos\theta\sin\theta)^2 + \sin^6\theta = 1$ (5)

i.e. $\cos^6\theta + \sin^6\theta = 1 - \frac{3}{8}(2\sin^2 2\theta)$ (5)

i.e. $\cos^6\theta + \sin^6\theta = 1 - \frac{3}{8}(1 - \cos 4\theta)$

i.e. $\cos^6\theta + \sin^6\theta = \frac{5}{8} + \frac{3}{8}\cos 4\theta$ (5)

Thus, We have

i.e. $\cos^6\theta + \sin^6\theta = a + b\cos 4\theta$, Where $a = \frac{5}{8}$ and (5)

$b = \frac{3}{8}$ (5)

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$\cos^6\theta + \sin^6\theta = (\cos^2\theta)^3 + (\sin^2\theta)^3$ (5)

$= (\cos^2\theta + \sin^2\theta)(\cos^4\theta - \cos^2\theta\sin^2\theta + \sin^4\theta)$

(10)

$= 1\left[(\cos^2\theta + \sin^2\theta)^2 - 3\cos^2\theta\sin^2\theta\right]$

$1\left[1 - 3\cos^2\theta\sin^2\theta\right]$

$= 1 - 3\left(\frac{\cos 2\theta + 1}{2}\right)\left(\frac{1 - \cos 2\theta}{2}\right)$ (5)

$1 - \frac{3}{4}(1 - \cos^2 2\theta)$

$= 1 - \frac{3}{4}\left[1 - \frac{(\cos 4\theta + 1)}{2}\right]$ (5)

$= 1 - \frac{3}{8}[2 - \cos 4\theta - 1]$

$= 1 - \frac{3}{8}(1 - \cos 4\theta)$

$= 1 - \frac{3}{8} + \frac{3}{8}\cos 4\theta$

$\cos^6\theta + \sin^6\theta = \frac{5}{8} + \frac{3}{8}\cos 4\theta$ (5)

$= a + b\cos 4\theta$

Where $a = \frac{5}{8}$ *and* $b = \frac{3}{8}$ (5) (5)

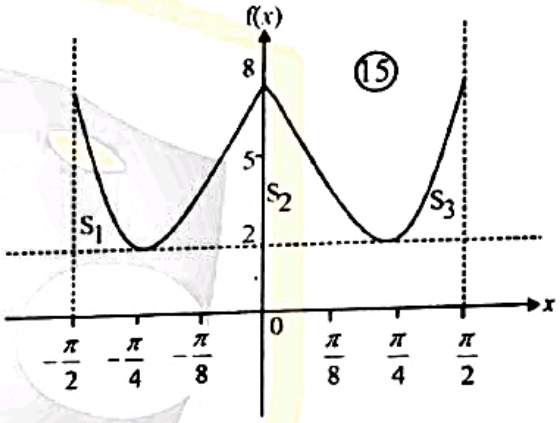
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(i) $f(x) = 8(\sin^6 x + \cos^6 x) = 8\left(\frac{5}{8} + \frac{3}{8}\cos 4x\right) = 5 + 3\cos 4x$ (5)

Drawing the graph of $f(x) = 8(\sin^6 x + \cos^6 x)$ in the range $|x| \le \frac{\pi}{2}$ is the same as drawing the graph of $f(x) = 5 + 3\cos 4x$ in the range $|x| \le \frac{\pi}{2}$

f(x)	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$5+3\cos 4x$	8	2	5	8	5	2	8

(15)



(15)

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(ii) $\cos^6 x + \sin^6 x = \frac{5}{4} + \frac{1}{2}\sin 4x$

Since $\cos^6 x + \sin^6 x = \frac{5}{8} + \frac{3}{8}\cos 4x$ *We have*

$\frac{5}{8} + \frac{3}{8}\cos 4x = \frac{5}{4} + \frac{1}{2}\sin 4x$ (5)

i.e. $3\cos 4x - 4\sin 4x = 5$

i.e. $3\cos 4x - 4\sin 4x = 5$

i.e. $\frac{3}{5}\cos 4x - \frac{4}{5}\sin 4x = 1$ (5)

i.e. $\cos\alpha\cos 4x - \sin\alpha\sin 4x = 1$. Where $\cos\alpha = \frac{3}{5}$ (5)

and $\sin\alpha = \frac{4}{5}$

i.e. $\cos(4x + \alpha) = 1$ (5)

$\Rightarrow 4x + \alpha = 2n\pi, \quad n \in \mathbb{Z}$

i.e. $x = \frac{n\pi}{2} - \frac{\alpha}{4}, n \in \mathbb{Z}$ (5)

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(b) Let $\alpha = \tan^{-1}\left(\frac{x-1}{x-2}\right)$ and $\beta = \tan^{-1}\left(\frac{x+1}{x+2}\right)$ (5)

Then $\tan\alpha = \frac{x-1}{x-2}$, $\tan\beta = \frac{x+1}{x+2}$ and $\alpha+\beta = \frac{\pi}{4}$ (5)

$\alpha+\beta = \frac{\pi}{4} \Rightarrow \tan(\alpha+\beta) = 1$ (5)

But $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ (5)

$$= \frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}$$ (5)

$$= \frac{(x-1)(x+2)+(x+1)(x-2)}{\left(x^2-4\right)-\left(x^2-1\right)}$$ (10)

$$= \frac{2x^2-4}{-3}$$ (5)

Thus, We have $\frac{2x^2-4}{-3} = 1 \Rightarrow x^2 = \frac{1}{2}$ (5)

$x = \pm\frac{1}{\sqrt{2}}$ (5) 50