Combined Mathematics I – 2010 Advanced Level Examination

❖ Answer six questions only.

01. (a) α and β are the roots of the quadratic equation $f(x) \equiv x^2 + px + q = 0$, where p and q are real and $2p^2 + q \neq 0$. If $y(p - x) = p + x$, substituting for x in $f(x) = 0$ or otherwise, show that $g(y) \equiv (2p^2 + q)y^2 + 2(q - p^2)y + q = 0$, where $y \neq -1$.

Hence, find the roots of the equation $g(y) = 0$ in terms of α and β.

Express $\left(\frac{\alpha}{2\beta+\alpha}\right)^2 + \left(\frac{\beta}{2\alpha+\beta}\right)^2$ in terms of p and q.

(b) If a, b, c and m are constants such that $a + b + c = 0$ and $ab + bc + ca + 3m = 0$, prove that $(y + ax)(y + bx)(y + cx = y(y^2 - 3mx^2) + abcx^3$.

If $y = x^2 + m$, show that $(x^2 + ax + m)(x^2 + bx + m)(x^2 + cx + m) = x^6 + abcx^3 + m^3$.

If $g(x) = x^6 + 16x^3 + 64$ has factors $(x^2 - 2x + m), (x^2 + ax + m)$ and $(x^2 + bx + m)$ find the values of m, a and b.

Hence,

(i) Show that $g(x)$ is non-negative for all x.

(ii) Find the roots of the equation $g(x) = 0$.

02. (a) Find how many different **four-digit** numbers can be formed from the seven digits $1, 2, 4, 5, 6, 8$ and 9, if any digit is selected.

(i) with repetition,

(ii) without repetition.

In case (i), find how many of the four-digit numbers do not have any digit repeated more than two times.

In case (ii), find how many of the four-digit numbers have two odd digits and two even digits. Find how many of them are even.

03. (b) For all $x \in \mathbb{R}$, let, in the usual notation,

$(1 + x)^n = {}^nC_0 + {}^nC_1x + \ldots.. + {}^nC_rx^r + \ldots\ldots {}^nC_nx^n$.

where n is a positive integer.

By considering the product of $(1+x)^{n-1}$ and $(1+x)$ show that

${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, for $r = 1, 2, \ldots\ldots\ldots, n-1$.

Deduce that ${}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 - \ldots..+(-1)^{n-1}\,{}^{n}C_{n-1} + (-1)^{n}\,{}^{n}C_n = 0$

Verify the above result by an **alternative method**.

If n is an even integer, deduce that ${}^{n}C_0 + {}^{n}C_2 + {}^{n}C_4 + \ldots..\,{}^{n}C_n = 2^{n-1}$.

03. By the Principle of Mathematical Induction prove that $4n^3 - 6n^2 + 4n - 1 = n^4 - (n-1)^4$ for any positive integer n.

Hence, write down U_r so that $U_r - U_{r-1} = 4r^3 - 6r^2 + 4r - 1$ for $r = 1, 2, \ldots\ldots\ldots$

Deduce that $\sum_{r=1}^{n} r^3 \left(\frac{n(n+1)}{2}\right)^2$

[You may assume that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$]

Write down V_r, the r^{th} term of the series

$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + (1^2 + 2^2 + 3^2 + 4^2) + \ldots\ldots$

Show that $\sum_{r=1}^{n} V_r = \frac{n(n+1)^2(n+2)}{12}$

Is this series convergent? Justify your answer. Let W_r be the r^{th} term of the series

$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \ldots\ldots\ldots$

Find $f(r)$ such that $W_r = f(r) - f(r+1)$.

Hence, find $S_n = \sum_{r=1}^{n} W_r$

In this series convergent? Justify your answer.

04. (a) Determine the locus of the complex number z which satisfies $|z-a| = |z+a|$, where a is a non-zero real number.

(b) Let z_1 and z_2 $(\neq 0)$ be two complex numbers such that $|z_1 - 2z_2| = |z_1 + 2z_2|$.

Using part (a) or otherwise, prove that $\frac{iz_1}{z_2} = k$ where k is real.

(i) Show that $|\arg(z_1) - \arg z_2| = \frac{\pi}{2}$.

(ii) The two points P_1 and P_2 in the Argand diagram represent the complex numbers $z_1 + 2z_2$ and $z_1 - 2z_2$ respectively.

If OP_1 is not perpendicular to OP_2 , show that $P_1\hat{O}P_2 = \tan^{-1}\left(\frac{4|k|}{k^2-4}\right)$, where O is the origin of the Argand plane.

If OP_1 is perpendicular to OP_2 determine the two possible values of k.

05. (a) Evaluate $\lim_{x\to 0}\frac{1-\cos 4x+x\sin 3x}{x^2}$.

(b) (i) Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $z = \tan^{-1}x$. Find $\frac{dy}{dz}$.

(ii) Let $y = e^{m\sin^{-1}x}$, Where m is a constant.

Show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$.

Find the value of $\frac{d^3y}{dx^3}$ at $x = 0$.

(c) A wire of given length l is cut into two portions. One portion is bent into the shape of a circle and the other portion into the shape of a square. Show that $A(x)$ the sum of the areas of the circle and the square is given by $A(x) = \frac{x^2}{4\pi} + \frac{(l-x)^2}{16}$ square units, where x, $(0 \le x \le l)$ is the length of the portion of the wire that is bent into the form of the circle.

Hence, show that the area $A(x)$ is minimum when the side of the square is equal to the diameter of the circle.

06. (a) Using partial fractions, find $\int \frac{2x}{(1+x^2)(1+x)^2}dx$

(b) Let $I = \int e^{ax}\cos bx\,dx$ and $J = \int e^{ax}\sin bx\,dx$, where a and b are non-zero real numbers.

Show that

(i) $bI + aJ = e^{ax}\sin bx$,

(ii) $aI - bJ = e^{ax}\cos bx$.

Hence, find I and J.

(c) By using the substitution $x^3t + 1 = 0$ or otherwise, show that $\int_{-1}^{-\frac{1}{2}} \frac{dx}{x(x^3-1)} = \frac{1}{3}\ln\left(\frac{9}{2}\right)$

07. (a) Show that the equations of the bisectors of the angle between the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are $\frac{a_1x+b_1y+c_1}{\sqrt{{a_1}^2+{b_1}^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{{a_2}^2+{b_2}^2}}$.

(b) The equation of a straight line through a point x_0, y_0, is given in the parametric form $\frac{x-x_0}{a} = \frac{y-y_0}{b} = t$, where $a^2 + b^2 = 1$ and t is a parameter.

Show that $|t|$ is the distance from the point (x_0, y_0) to the point (x, y) measured along the line.

(c) $ABCD$ is a rhombus that entirely lies in the first quadrant. The equations of AB and AD are $x - 2y + 5 = 0$ and $2x - y + 1 = 0$ respectively. The angle BAD is acute and $AC = 2\sqrt{2}$. Using parts (a) and (b) or otherwise, find the equations of AC, and the two remaining sides of the rhombus. If E is the point of intersection of the diagonals of the rhombus, find the length of DE and hence, find the area of the rhombus.

08. State the conditions for two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ to touch each other internally or externally.

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a circle and $P_1(x_1,\ y_1)$ be a point which lies outside the circle $S = 0$. Show that the length of a tangent drawn from the point P_1 to the circle $S = 0$ is given by $\sqrt{{x_1}^2 + {y_1}^2 + 2gx_1 + 2fy_1 + c}$.

Prove that the two circles $S_1 \equiv x^2 + y^2 + 4x - 2y - 5 = 0$ and $S_2 \equiv x^2 + y^2 - 8x - 6y + 15 = 0$ touch each other externally.

Find the coordinates of A, the point of contact of the two circles $S_1 = 0$ and $S_2 = 0$.

Let P be a point such that the length of a tangent from the point P to the circle $S_1 = 0$ is equal to k times the length of a tangent from the point P to the circle $S_2 = 0$.

Prove that the locus of the point P,

(i) if $k = 1$, is a line through the point A perpendicular to the line joining the centres of the two circles $S_1 = 0$ and $S_2 = 0$.

(ii) If $k \neq 1$, is a circle through the point A.

Write down the equation of the locus of P when $k = \frac{1}{2}$ and show that if touches one of the two circles $S_1 = 0$ and $S_2 = 0$ externally and the other internally at the point A.

09. (a) State and prove the Cosine rule for a triangle ABC, in the usual notation.

In the usual notation for a triangle ABC, show that

(i) $2\left(\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}\right) = \frac{a^2+b^2+c^2}{abc}$,

(ii) if $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$, then the angle C is $\frac{\pi}{3}$.

(b) Express $\sqrt{3}\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$ where R and α are real.

Hence, find the general solution of the equation

$\sqrt{3}cos^2\theta + \left(1 - \sqrt{3}\right)\sin\theta\cos\theta - sin^2\theta - \cos\theta + \sin\theta = 0$.

(c) Show that $\cos^{-1}(-x) = \pi - \cos^{-1}x$ for $-1 \leq x \leq 1$.