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G.C.E. (Advanced Level) Examination - August 2010
Combined Mathematics I
Scheme of Marking

(01) $y(p - x) = p + x \Rightarrow x = \dfrac{p(y-1)}{y+1}$ if $y \neq -1$

Substituting for x in f(x) = 0 wehave

$f(x) = x^2 + px + q = \left\{\dfrac{p(y-1)}{y+1}\right\}^2 + p\left(\dfrac{p(y-1)}{y+1}\right) + q = 0$ (05)

i.e $p^2(y-1)^2 + p^2(y^2 - 1) + q(y+1)^2 = 0$

$g(y) = (2p^2 + q)y^2 + 2(q - p)^2 y + q = 0$ (05) [10]

Since α and β are the roots of $f(x) \equiv x^2 + px + q = 0$,

We have $\alpha + \beta = -p$ and $\alpha\beta = q$ (05)

If $x = \alpha$ then, $y = \dfrac{p+\alpha}{p-\alpha} = \dfrac{-\beta}{-2\alpha-\beta} = \dfrac{\beta}{2\alpha+\beta}$ (05)

Since α is a root of $x^2 + px + q = 0$ it satisfies the equation $x^2 + px + q = 0$

Thus $y = \dfrac{\beta}{2\alpha+\beta}$ Satisfies the equation $g(y) = 0$

Hence, $\dfrac{\beta}{2\alpha+\beta}$ is a root of $g(y) = 0$ (05)

Similarly, When $x = \beta$ We have $y = \dfrac{\alpha}{2\beta+\alpha}$ (05)

By a similar argument above we can show that $\dfrac{\alpha}{2\beta+\alpha}$ is a root of $g(y) = 0$ (05)

Since $g(y) = 0$ is a quadratic equation $\dfrac{\alpha}{2\beta+\alpha}$ and $\dfrac{\beta}{2\alpha+\beta}$ are the roots of $f(y) = 0$ [25]

$$\left(\frac{\alpha}{2\beta+\alpha}\right)^2 + \left(\frac{\beta}{2\alpha+\beta}\right)^2 = \left[\left\{\frac{\alpha}{2\beta+\alpha} + \frac{\beta}{2\alpha+\beta}\right\}^2 - 2\left(\frac{\alpha}{2\beta+\alpha}\right)\left(\frac{\beta}{2\alpha+\beta}\right)\right] (05)$$

$$= \left\{\frac{-2(q-p^2)}{2p^2+q}\right\}^2 - 2\left(\frac{q}{2p^2+q}\right) (05)$$

$$= \frac{4(q^2 - 2qp^2 + p^4) - 2q(2p^2+q)}{(2p^2+q)^2}$$

$$= \frac{2(q^2 - 6qp^2 + 2p^4)}{(2p^2+q)^2}$$ (05) [15]

(b) $(y + ax)(y + bx)(y + cx)$

$= y^3 + (a + b + c)xy^2 + (ab + bc + ca)x^2y + abcx^3$ (05)

$= y^3 - 3mx^2y + abc\,x^3$ (05)

$= y(y^2 - 3mx^2) + abcx^3$ [10]

$= (x^2 + m)\{(x^2 + m)^2 - 3mx^2\} + abcx^3$

(writing $y = x^2 + m$)

$= (x^2 + m)(x^4 - mx^2 + m^2) + abc\,x^3 = x^6 + abcx^3 + m^3$ (05)

i.e $(x^2 + ax + m)(x^2 + bx + m)(x^2 + cx + m) = x^6 + abcx^3 + m^3$ → (01) [05]

If $g(x) = x^6 + 16x^3 + 64$ has factors $(x^2 - 2x + m)$, $(x^2 + ax + m)$ and $(x^2 + bx + m)$ We can write

$x^6 + 16x^3 + 64 = (x^2 - 2x + m)(x^2 + ax + m)(x^2 + bx + m)$

Using Eqn. (01) above We have $m^3 = 64$ and $-2ab = 16$

$\Rightarrow$

$m = 4$ and $ab = -8$

Also We have $-2 + a - b = 0$ (05)

$(a - b)^2 = (a+b)^2 - 4ab = 4 + 32 = 36$ (05) $\Rightarrow a - b = \pm 6$

Thus We have $a = -2$ and $b = 4$ or $a = 4$ and $b = -2$

(05) (05) [20]

(i) $g(x) = (x^2 - 2x + 4)^2(x^2 + 4x + 4)$

$= (x^2 - 2x + 4)^2(x + 2)^2 \geq 0$ for all x

(05)

[05]

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(ii) $g(x) = (X^2 - 2X + 4)^2 (X + 2)^2 = 0 \Rightarrow X = -2,$
$-2, 1 \pm \sqrt{3}\, i,\ 1 \pm \sqrt{3}\, i$ (10) [10]

(02) (a)

(i) The number of different four – digit numbers that can be formed from seven digits with repetitions
$= 7 \times 7 \times 7 \times 7 = 2401$ (05) [05]

(ii) The number of different four – digit numbers that can be formed from seven digits without repetitions
$= 7 \times 6 \times 5 \times 4 = 840$ (05) [05]

In case (i) :

The number of different four – digit numbers that have the same digit repeated four times = 7 (05)

The number of different four – digit numbers that have the same digit repeated three times and any other digit once
$= 7 \times 6 = 42$

Each such four – digit number can be arranged in $\frac{4!}{3!} = 4$ ways

Thus the total number of such four – digit numbers
$= 42 \times 4 = 168$ (05)

The total number of different four – digit numbers that do not have any digit repeated more than two times
$= 2401 - (168 + 7) = 2226$
(05) (05) [20]

Other :

The number of different four – digit numbers that have four different digits $= 7 \times 6 \times 5 \times 4 = 840$ (05)

The number of different four – digit numbers with one digit repeated twice and the other two digits different from each other $= {}^7C_1 \times {}^6C_2 = 7 \times 15 = 105$

Each such number can be arranged in $\frac{4!}{2!} = 12$ ways

Thus the total number of such four - digit numbers
$= 105 \times 12 = 1260$ (05)

The number of different four - digit numbers with two different digits each repeated twice $= {}^7C_2 = 21$

Each such number can be arranged in $\frac{4!}{2!\ 2!} = 6$ ways

Thus the total number of such four - digit numbers
$= 21 \times 6 = 126$ (05)

The total number of different four - digit numbers that do not have any digit repeated more than two times
$= 840 + 1260 + 126 = 2226$ (05) [20]

In cose (ii)

The number of four - digit numbers with two odd and two even digits $= {}^3C_2 \times {}^4C_2$ (05)

The total number of arrangements of each selection = 4!
(05)

Thus the total number of different four - digit numbers with two odd and two even digits $= {}^3C_2 \times {}^4C_2 \times 4!$
$= 3 \times 6 \times 24$
$= 432$ (05) [15]

For a number to be even the last digit should be an even digit

Thus the number of four - digit numbers with the last digit even $= {}^4C_1$ (05)

The number of selections of two odd digits and one even digit $= {}^3C_2 \times {}^3C_1$ (05)

The total nunber of arrangements of each selection = 3!

Thus the total number of four - digit even numbers that have two odd digits and two even digits
$= {}^3C_2 \times {}^3C_1 \times {}^4C_1 \times 3! = 3 \times 3 \times 4 \times 6 = 216$ (05) [15]

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(b) ${}^nC_0 + {}^nC_1x + \ldots\ldots + {}^nC_rx^r + \ldots\ldots {}^nC_nx^n$

$= (1+x)^n$

$= (1+x)(1+x)^{n-1}$

$= (1+x)\left({}^{n-1}C_0 + {}^{n-1}C_1x + \ldots\ldots + {}^{n-1}C_rx^r + \ldots. + {}^{n-1}C_{n-1}x^{n-1}\right)$

$= \left[\begin{matrix} {}^{n-1}C_0\left({}^{n-1}C_0 + {}^{n-1}C_1\right)x + \ldots\ldots + \\ \left({}^{n-1}C_{r-1} + {}^{n-1}C_r\right)x^r + \ldots.. + {}^{n-1}C_{n-1}x^n \end{matrix}\right]$ (05)

Comparing the coefficients of corresponding terms we have

${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, for r = 1, 2, ..., n - 1 (05)

10

${}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots\ldots + (-1)^{n-1}\ {}^nC_{n-1} + (-1)^n\ {}^nC_n$

$= \left[\begin{matrix} {}^nC_0 - \left({}^{n-1}C_0 + {}^{n-1}C_1\right) + \left({}^{n-1}C_1 + {}^{n-1}C_2\right) - \ldots\ldots + \\ (-1)^{n-1}\left({}^{n-1}C_{n-2} + {}^{n-1}C_{n-1}\right) + (-1)^n\ {}^nC_n \end{matrix}\right]$ (05) (05)

$= 1 - 1 + (-1)^{n-1} + (-1)^n = 0$ (05)

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Verification :

Putting x = -1 in

$(1+x)^n = {}^nC_0 + {}^nC_1x + \ldots\ldots + {}^nC_rx^r + \ldots\ldots + {}^nC_nx^n$

we have

$(1-1)^n = \left(\begin{matrix} {}^nC_0 + {}^nC_1(-1) + \ldots\ldots + {}^nC_r(-1)^r \\ + \ldots\ldots + {}^nC_{n-1}(-1)^{n-1} + {}^nC_n(-1)^n \end{matrix}\right)$

i.e. $\left(\begin{matrix} {}^nC_0 - {}^nC_1 + {}^nC_2 + \ldots\ldots + (-1)^{n-1} \\ {}^nC_{n-1} + (-1)^n\ {}^nC_n = 0 \end{matrix}\right)$ (05)

05

If n is an even integer we have

${}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots\ldots - {}^nC_{n-1} + {}^nC_n = 0 \rightarrow (01)$

Putting x = 1 in

$(1+x)^n = {}^nC_0 + {}^nC_1x + \ldots\ldots + {}^nC_rx^r + \ldots\ldots + {}^nC_nx^n$

We have

$(1+1)^n = \left(\begin{matrix} {}^nC_0 + {}^nC_1 + \ldots.. + {}^nC_r \\ + \ldots\ldots + {}^nC_{n-1} + {}^nC_n \end{matrix}\right) \rightarrow (02)$ (05)

$(01) + (02) \Rightarrow {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots + {}^nC_n = 2^{n-1}$ (05)

10

(03) $4n^3 - 6n^2 + 4n - 1 = n^4 - (n-)^4 \rightarrow (01)$

When n = 1, L . H. S. of (01) = 1 = R. H . S of (01)

Thus the result is true for n = 1 (05)

Assume that the result is true for n = p

i. e. assume that $4p^3 - 6p^2 + 4p - 1 = p^4 - (p-1)^4$ (05)

Suppose that n = p + 1

$4(p+1)^3 - 6(p+1)^2 + 4(p+1) - 1$

$= 4(p^3 + 3p^2 + 3p + 1) - 6(p^2 + 2p + 1) + 4(p+1) - 1$ (05)

$= 4p^3 + 6p^2 + 4p + 1 = p^4 + 4p^3 + 6p^2 + 4p + 1 - p^4$

$= (p+1)^4 - p^4$ (05)

Thus the result is true for n = p + 1

By the principle of Mathematical Induction result is true for any positive integer n. (05)

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Using first part we can write

$4r^3 - 6r^2 + 4r - 1 = r^4 - (r-1)^4 = u_r - u_{r-1}$ where $u_r = r^4$ (05)

05

When r = 1, $u_1 - u_0 = 4.1^3 - 6.1^2 + 4.1 - 1$

When r = 2, $u_2 - u_1 = 4.2^3 - 6.2^2 + 4.2 - 1$

....

.... (05)

....

When r = n-1, $u_{n-1} - u_{n-2} = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$

When r = n, $u_n - u_{n-1} = 4.n^3 - 6.n^2 + 4.n - 1$ +

$u_n - u_0 = 4\sum_{r=1}^{n} r^3 - 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - n$ (05)

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$$4\sum_{r=1}^{n} r^3 = +6\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + n + u_n - u_0$$

$$= 6\frac{n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n + n^4$$ (05)

$$= n(n+1)\left(2n+1-2+n^2-n+1\right)$$

$$= n^2(n+1)^2$$ (05)

$$\sum_{r=1}^{n} r^3 = -\left(\frac{n(n+1)}{2}\right)^2$$ [20]

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+(1^2+2^2+3^2+4^2)+\ldots\ldots$$

$V_r = 1^2+2^2+\ldots\ldots+r^2 = \frac{r(r+1)(2r+1)}{6}$ (05)

$$= \frac{1}{3}r^3 + \frac{1}{2}r^2 + \frac{1}{6}r$$ [05]

$$\sum_{r=1}^{n} v_r = \sum_{r=1}^{n} \frac{r(r+1)(2r+1)}{6}$$

$$= \frac{1}{3}\sum_{r=1}^{n} r^3 + \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{6}\sum_{r=1}^{n} r$$ (05)

$$= \left(\begin{array}{l} \frac{1}{3}\left\{\frac{n(n+1)}{2}\right\}^2 + \frac{1}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} \\ + \frac{1}{6}\left\{\frac{n(n+1)}{2}\right\} \end{array}\right)$$ (05)

$$= \frac{n(n+1)}{12}\left(n^2+n+2n+1+1\right)$$

$$= \frac{n(n+1)}{12}\left(n^2+3n+2\right) = \frac{n(n+1)^2(n+2)}{12}$$ (05)

[15]

$$\lim_{n\to\infty} \sum_{r=1}^{n} v_r = \lim_{n\to\infty} \frac{n(n+1)^2(n+2)}{12} = \infty$$

Since the sum to infinity is not finite the series is not convergent (05) [05]

$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \ldots\ldots$$

$$W_r = \frac{2r+1}{1^2+2^2+\ldots\ldots+r^2}$$ (05)

$$= \frac{2r+1}{\left\{\frac{r(r+1)(2r+1)}{6}\right\}} = \frac{6}{r(r+1)} = \frac{6}{r} - \frac{6}{r+1}$$

$= f(r) - f(r+1)$ *where*

$f(r) = \frac{6}{r}$ (05) [10]

When r = 1, $W_1 = f_{(1)} - f_{(2)}$

When r = 2, $W_2 = f_{(2)} - f_{(3)}$

.....

.....

..... (05)

When r = n-1, $W_{n-1} = f_{(n-1)} - f_{(n)}$

When r = n, $W_n = f_{(n)} - f_{(n+1)}$ +

$$S_n = \sum_{r=1}^{n} w_r = f_{(1)} - f_{(n+1)}$$

$$= 6 - \frac{6}{n+1}$$ (05) [10]

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[6 - \frac{6}{n+1}\right]$$

$$= 6 - \lim_{n\to\infty} \frac{6}{n+1} = 6 \quad \therefore \lim_{n\to\infty} \frac{6}{n+1} = 0$$

Since the sum to infinity is finite the series is convergent.

(05) [05]

(04) (a) Let z = x + iy

Then, z - a = (x - a) + iy and z + a = (x + a) + iy

$|z-a|=\sqrt{(x-a)^2+y^2}$ and $|z+a|=\sqrt{(x+a)^2+y^2}$ (05)

$|z-a|=|z+a| \Rightarrow \sqrt{(x-a)^2+y^2}$

$=\sqrt{(x+a)^2+y^2}$

$(x-a)^2+y^2=(x+a)^2+y^2 \Rightarrow 4ax=0 \Rightarrow x=0$ (05)

Thus the locus of the coplex number Z which satisfies |Z - a| = |Z + a| is imaginary axis of the Argand plane. (05)

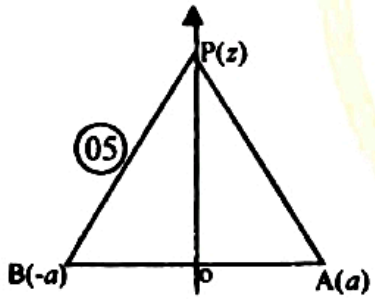
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Aliter :

Let p be the point in the Argand diagram the represents the complex number z.

AP = |z - a| and BP = |z + a| where A and B are the points in the Argand diagram the represents the numbers a and $-a$.

$|z_1 - a| = |z_1 + a| \Rightarrow AP = BP$ (05)



Thus the point P which satisfies the condition |z - a| = |z + a| varies on the perpendicular bisector of AB which is the imaginary axis of the Argand plane. (05)

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(b) $|z_1-2z_2|=|z_1+2z_2| \Rightarrow \left|\frac{z_1}{z_2}-2\right|=\left|\frac{z_1}{z_2}+2\right|$ (05)

By part (a) the point that represents the complex number $\frac{z_1}{z_2}$ lies on the imaginary axis of the Argand plane. (05)

Thus the point that represents the complex number $\frac{iz_1}{z_2}$ lies on the real axis of the Argand plane. (05)

The complex number $\frac{iz_1}{z_2}$ can be written as $\frac{iz_1}{z_2}=k$, where k is real. (05)

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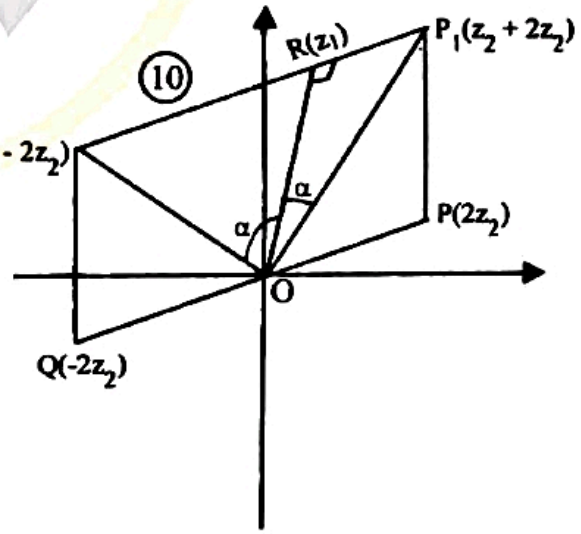
(i) Since the point that represents the complex number $\frac{z_1}{z_2}$ lies on the imaginary axis of the Argand plane we have $\arg\left(\frac{z_1}{z_2}\right)=\pm\frac{\pi}{2}$, Where positive value corresponding to a point on the positive side of the imaginary axis and negative value, corresponding to a point on the negative side of the imaginary axis. (10)

$\left|\arg\left(\frac{z_1}{z_2}\right)\right|=\frac{\pi}{2} \Rightarrow |\arg(z_1)-\arg(z_2)|=\frac{\pi}{2}$ (05)

$\because \arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$

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(ii)



$|z_1-2z_2|=|z_1+2z_2| \Rightarrow OP_2=OP_1$ (05)

Further,

$RP_1=OP=|2z_2|=|-2z_2|=OQ=RP_2, O\hat{R}P_1=\frac{\pi}{2}$ (05)

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$$\tan\alpha = \frac{RP_1}{OR} = \frac{OP}{OR} = \frac{|2z_2|}{|z_1|} = 2\left|\frac{z_2}{z_1}\right| = \frac{2}{|k|} \because \frac{iz_1}{z_2} = k$$

(05) (05)

If OP_1 is not perpendicular to OP_2 then $\alpha \neq \frac{\pi}{4}$

$\Rightarrow \tan\alpha \neq 1 \Rightarrow |k| \neq 2$ (05)

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{\frac{4}{|k|}}{1-\frac{4}{|k|^2}} = \frac{4|k|}{k^2-4}$$ (05)

i.e. $P_1\hat{O}P_2 = \tan^{-1}\left(\frac{4|k|}{k^2-4}\right)$ 40

If OP_1 is perpendicular to OP_2 then $\alpha = \frac{\pi}{4}$

$\Rightarrow \tan\alpha = \frac{2}{|k|} = 1 \Rightarrow |k| = 2 \Rightarrow k = \pm 2$ (10) 10

(05) (a) $\lim_{x\to o} \frac{1-\cos 4x + x\sin 3x}{x^2}$

$= \lim_{x\to o} \frac{1-\cos 4x}{x^2} + \lim_{x\to o} 3\left(\frac{\sin 3x}{3x}\right)$ (05)

$= \lim_{x\to o} \frac{2\sin^2 2x}{x^2} + 3\lim_{x\to o}\left(\frac{\sin 3x}{3x}\right)$

$= 8\left\{\lim_{x\to o} \frac{\sin 2x}{2x}\right\}^2 + 3 \quad \because \lim_{x\to o}\left(\frac{\sin 3x}{3x}\right) = 1$ (05)

$= 8 + 3 = 11 \quad \because \lim_{x\to o}\left(\frac{\sin 2x}{2x}\right) = 1$ (05) 15

(b) (i) $z = \tan^{-1} x \Rightarrow x = \tan z$

Substituting $x = \tan z$ in $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

we have $y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2 z}-1}{\tan z}\right)$ (05)

$= \tan^{-1}\left(\frac{\sec z - 1}{\tan z}\right) = \tan^{-1}\left(\frac{1-\cos z}{\sin z}\right)$ (05)

$= \tan^{-1}\left[\frac{2\sin^2\left(\frac{z}{2}\right)}{2\sin\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right)}\right]$ (05)

$= \tan^{-1}\left[\tan\left(\frac{z}{2}\right)\right] = \frac{z}{2}$ (05)

$\frac{dy}{dz} = \frac{1}{2}$ (05) 25

Aliter : $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

$$\frac{dy}{dx} = \frac{1}{1+\left[\frac{\sqrt{1+x^2}-1}{x}\right]^2} \times \frac{x\cdot\frac{1}{2\sqrt{1+x^2}}\cdot 2x - \left\{\sqrt{1+x^2}-1\right\}}{x^2}$$ (05) (05)

$$= \frac{1}{2\left(1+x^2-\sqrt{1+x^2}\right)} \cdot \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}} = \frac{1}{2\left(1+x^2\right)}$$ (05)

$z = \tan^{-1}(x) \Rightarrow \frac{dz}{dx} = \frac{1}{1+x^2}$ (05)

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = \frac{1}{2\left(1+x^2\right)} \cdot \frac{\left(1+x^2\right)}{1} = \frac{1}{2}$$ (05) 25

(ii) $y = e^{m\sin^{-1}x}$

Differentiating with respect to x we have

$\frac{dy}{dx} = e^{m\sin^{-1}x} \times \frac{m}{\sqrt{1-x^2}}$ (05)

$\sqrt{1-x^2}\ \frac{dy}{dx} = my$ (05)

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Differentiating above equation with respectter x we have

$$\sqrt{1-x^2}\ \frac{d^2y}{dx^2} - \frac{2x}{2\sqrt{1-x^2}}\ \frac{dy}{dx} = m\ \frac{dy}{dx} \quad (05)$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m\sqrt{1-x^2}\ \frac{dy}{dx}\ m^2y \quad (05)$$

$$i.e.\ \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0 \quad \boxed{20}$$

Differentiating this equation with respect to x we have

$$\left(1-x^2\right)\frac{d^3y}{dx^3} - 2x\frac{d^2y}{dx^2} - x\frac{d^2y}{dx^2} - \frac{dy}{dx} - m^2\frac{dy}{dx} = 0 \quad (05)$$

$$\left(1-x^2\right)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} - \left(1+m^2\right)\frac{dy}{dx} = 0$$

When $x = 0$ We have $y = 1$.

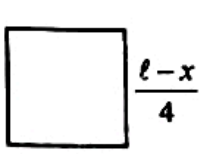
$$\frac{dy}{dx} = m \quad (05) \quad and \quad \frac{d^3y}{dx^3} = \left(1+m^2\right)m \quad (05) \quad \boxed{15}$$

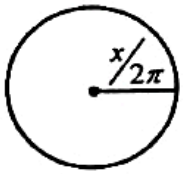
(c) The radius of the circle is $\frac{x}{2\pi}$ and the length of a side of the square is $\frac{\ell - x}{4}$ (05)

Thus the area of the circle $= \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$ sq. units

and the area of the square $= \frac{(\ell - x)^2}{16}$ sq. units

$$A(x) = \frac{x^2}{4\pi} + \frac{(\ell - x)^2}{16}\ sq.\ units \quad (05)$$





$\boxed{10}$

$$\frac{dA(x)}{dx} = \frac{x}{2\pi} - \frac{(\ell - x)}{8} = \frac{(4+\pi)x - \ell\pi}{8\pi} = \left(\frac{4+\pi}{8\pi}\right)\left(x - \frac{\ell\pi}{4+\pi}\right)$$ (05)

$$\frac{dA(x)}{dx} = 0 \Rightarrow \left(x - \frac{\ell\pi}{4+\pi}\right) = 0 \Rightarrow x = \frac{\ell\pi}{4+\pi} \quad (05)$$

When

$$x < \frac{\ell\pi}{4+\pi},\ \frac{dA(x)}{dx} < 0\ and\ when\ x > \frac{\ell\pi}{4+\pi},\ \frac{dA(x)}{dx} > 0$$

$$\Rightarrow A(x)\ is\ minimum\ at\ x = \frac{\ell\pi}{4+\pi}$$

When $x = \frac{\ell\pi}{4+\pi}$ the diameter of the circle is $\frac{\ell}{4+\pi}$ and the length of a side of the square is

$$\frac{1}{4}\left(\ell - \frac{\ell\pi}{4+\pi}\right) = \frac{1}{4+\pi} \quad (05)$$

Hence we have the result. $\boxed{15}$

(06) (a) $$\frac{2x}{\left(1+x^2\right)(1+x)^2} \equiv \frac{Ax+B}{1+x^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2} \quad (05)$$

$$\begin{aligned} 2x &= (Ax+B)(1+x)^2 + C(1+x)(1+x^2) + D(1+x^2) \\ &= (A+C)x^3 + (2A+B+C+D)x^2 + (A+2B+C)x \\ &\quad + (B+C+D) \end{aligned}$$

Comparing the coefficients of corresponding terms

We have

$A + C = 0$, $2A + B + C + D = 0$, $A + 2B + C = 2$ and $B + C + D = 0$ (05)

(05) (05)

Solving the above set of equation we have $A = C = 0$, $B = 1$ and $D = -1$ (05) (05)

$$\int \frac{2x}{\left(1+x^2\right)(1+x)^2}dx = \int \frac{dx}{1+x^2} - \int \frac{dx}{(1+x)^2}$$

$$= \tan^{-1}(x) + \frac{1}{1+x} + k, \quad (05)\ (05)$$

Where k is an arbitrary constant. $\boxed{40}$

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(b) (i) $bI = b\int e^{ax}\cos bx\,dx$

$= e^{ax}\sin bx - a\int e^{ax}\sin bx\,dx$ (05) (05)

$= e^{ax}\sin bx - aJ$

i.e. $bI + aJ = e^{ax}\sin bx \rightarrow (1)$ [10]

(ii) $-bJ = -b\int e^{ax}\sin bx\,dx$

$= e^{ax}\cos bx - a\int e^{ax}\cos bx\,dx$ (05) (05)

$= e^{ax}\cos bx - a\underline{I}$

i.e. $a\underline{I} - bJ = e^{ax}\cos bx \rightarrow (2)$ [10]

(b) x (1) + a x (2) $\Rightarrow (a^2 + b^2)\,\underline{I} = e^{ax}(b\sin bx + a\cos bx)$

i.e. $\underline{I} = \dfrac{e^{ax}}{(a^2+b^2)}(b\sin bx + a\cos bx)$ (05) [05]

$a\times(01) - b\times(02) \Rightarrow (a^2+b^2)J = e^{ax}(a\sin bx - b\cos bx)$

i.e. $J = \dfrac{e^{ax}}{(a^2+b^2)}(a\sin bx - b\cos bx)$ (05) [05]

(c) $x^3 t + 1 = 0 \Rightarrow x^3 = -\dfrac{1}{t}$

$3x^2 t + x^3\dfrac{dt}{dx} = 0 \Rightarrow \dfrac{dx}{x} = -\dfrac{dt}{3t}$, (05)

When $x = -1,\ t = 1$

and when $x = -\dfrac{1}{2},\ t = 8$ (05)

$$\int_{-1}^{-1/2}\frac{dx}{x(x^3-1)} = \int_1^8 \frac{\frac{-dt}{3t}}{\left(\frac{-1}{t}-1\right)} = \frac{1}{3}\int_1^8\frac{dt}{1+t}$$ (05) (05)

$= \frac{1}{3}\left[\ell n|1+t|\right]_1^8$ (05)

$= \frac{1}{3}(\ell n\,9 - \ell n\,2)$ (05)

$= \frac{1}{3}\ell n\,\frac{9}{2}$ [30]

(07) (a) Let $P_0(x_0, y_0)$ be any point on the bisector of the angle between the two giving lines.

$P_0P_1 = \dfrac{|a_1x_0 + b_1y_0 + c_1|}{\sqrt{a_1^2+b_1^2}}$ *and* $P_0P_2 = \dfrac{|a_2x_0 + b_2y_0 + c_2|}{\sqrt{a_2^2+b_2^2}}$ (05)

$P_0P_1 = P_0P_2 \Rightarrow \dfrac{a_1x_0 + b_1y_0 + c_1}{\sqrt{a_1^2+b_1^2}} = \pm\dfrac{a_2x_0 + b_2y_0 + c_2}{\sqrt{a_2^2+b_2^2}}$ (05)

since the point $P_0(x_0, y_0)$ be any point on a bisector we have the equations of the bisectors as

$\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2+b_1^2}} = \pm\dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2+b_2^2}}$ (05) [15]

(b) The distance between the point (x, y) and (x_0, y_0)

$= \sqrt{(x-x_0)^2 + (y-y_0)^2} = \sqrt{(at)^2 + (bt)^2}$ (05)

$= \sqrt{(a^2+b^2)t^2} = |t|$ (05)

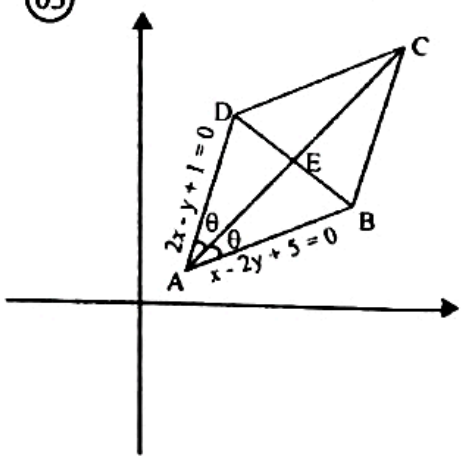
$\therefore a^2 + b^2 = 1$ [10]

(c) Using part (*a*) We write the equations of the bisectors of the angle BAD as

$\dfrac{x - 2y + 5}{\sqrt{1+4}} = \pm\dfrac{2x - y + 1}{\sqrt{4+1}}$ (05)

i. e. $x + y - 4 = 0$ and $y - x - 2 = 0$ (05)

(05)



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AC is one of the bisector and it is corresponding to the acute angle of BAD. Thus the angle between AC and AB or AD should be less than $\frac{\pi}{4}$ (05)

Let this angle be θ

By considering AD and the line given by $y - x - 2 = 0$

We have, $\tan\theta = \left|\frac{2-1}{2+1}\right| = \frac{1}{3} < 1$ (05), Thus $y - x - 2 = 0$ is the equation corresponding to the line AC. (05) [30]

Solving $y - x - 2 = 0$ and $2x - y + 1 = 0$ We have A = (1, 3) using part (b) we write the equations of AC as (05)

$\frac{y^1 - 3}{1/\sqrt{2}} = \frac{x^1 - 1}{1/\sqrt{2}} = |2\sqrt{2}|$ (05), where (x^1y^1) is a point AC at a distance $2\sqrt{2}$ from A.

The possible pairs of (x^1y^1) are (3, 5) and (-1, 1) since the rhombus entirely lies in the first quadrant we have C = (3, 5) (05)

BC is parallel to AD and hence We can write the equation of BC as $2x - y + \lambda = 0$, Where λ is a parameter.

Since BC passes through (3, 5) we have $\lambda = -1$

Thus the equation of BC is $2x - y - 1 = 0$ (05) [20]

DC is parallel to AB and hence we can write the equation of DC as $x - 2y + \mu = 0$, Where μ is a parameter.

Since DC passes through (3, 5) we have $\mu = 7$

Thus the equation of DC is $x - 2y + 7 = 0$ (05) [05]

DE = AE $\tan\theta = \sqrt{2} \cdot \frac{1}{3} = \frac{\sqrt{2}}{3}$ (05) (05) [10]

The area of ABCD $= 4 \times$ *area* of AED

$= 4 \times \frac{1}{2} \cdot DE \cdot AE$

$= 4 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{3} \cdot \sqrt{2}$ (05)

$= \frac{4}{3}$ *sq. units* (05) [10]

(08) Let d be the distance between the centres of the circles.

Then $d = \sqrt{(g_1 - g_2)^2 + (f_1 - f_2)^2}$

Let r_1 and r_2 be the radii of the two circles.

Then, $r_1 = \sqrt{g_1^2 + f_1^2 - c_1}$ *and* $r_2 = \sqrt{g_2^2 + f_2^2 - c_2}$

The condition for two circles to touch each other internally is $d = |r_1 - r_2|$ (05)

The condition for two circles to touch each other externally is $d = r_1 + r_1$ (05) [10]

$(P_1C)^2 = (x_1 + g)^2 + (y_1 + f)^2$

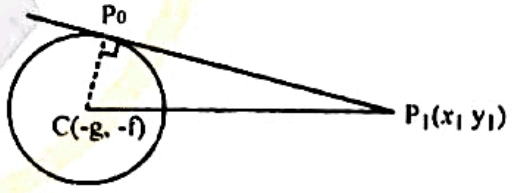
$(P_0C)^2 = g^2 + f^2 - C$

$(P_0P_1)^2 = (P_1C)^2 - (P_0C)^2$

$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - C)$ (05)

$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\therefore P_0P_1 = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ (05)



[10]

$S_1 \equiv x^2 + y^2 + 4x - 2y - 5 = 0 \Rightarrow C_1 = (-2, 1)$ and $r_1 = \sqrt{4 + 1 + 5} = \sqrt{10}$ (05)

$S_2 \equiv x^2 + y^2 - 8x - 6y + 15 = 0 \Rightarrow C_2 = (4, 3)$ and $r_2 = \sqrt{16 + 9 - 15} = \sqrt{10}$ (05)

$C_1C_2 = \sqrt{(4 + 2)^2 + (3 - 1)^2} = 2\sqrt{10} = r_1 + r_2$ (05)

Thus the two circles $S_1 = 0$ and $S_2 = 0$ touch each other externally.

[15]

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Since the point A is the midpoint of $C_1 C_2$ we have A = (1, 2) (05) [05]

Let $P = (x^1, y^1)$

Then the length of a tangent drawn from P to the circle $S_1 = 0$ is $\sqrt{x^{12} + y^{12} + 4x^1 - 2y^1 - 5}$

The length of a tangent drawn from P to the circle $S_2 = 0$ is $\sqrt{x^{12} + y^{12} - 8x^1 - 6y^1 + 15}$

$x^{12} + y^{12} + 4x^1 - 2y^1 - 5 = k^2 (x^{12} + y^{12} - 8x^1 - 6y^1 + 15)$ (05)

$(1-k^2) x^{12} + (1-k^2) y^{12} + (4+8k^2) x^1 - (2-6k^2) y^1 - 5 - 15k^2 = 0$

(i) If k = 1, we have $12x^1 + 4y^1 - 20 = 0$

Thus the locus of the point P is the treaight line $3x + y - 5 = 0$ which has the gradient -3 (05)

The cordinates (1, 2) of the point A satisfies the equation $3x - y - 5 = 0$

The gradient of $C_1 C_2 = \frac{3-1}{4+2} = \frac{1}{3}$

(The gradient of $3x + y - 5 = 0$) (05)

(The gradient of $C_1C_2 = (-3)\left(\frac{1}{3}\right) = -1$

Thus the locus of P is the line through A perpendicular to C_1C_2 (05) [20]

(ii) If $k \neq 1$ then $1 - k^2 \neq 0$

Thus the locus of the point P is a circle given by the equation

$(1-k^2)x^2 + (1-k^2) y^2 + (4+8k^2) x - (2-6k^2) y - 5 - 15k^2 = 0$ (05)

Substituting the coordinates (1, 2) in the above equation we have,

$(1-k^2)1^2 + (1-k^2) 2^2 + (4+8k^2) (1) - (2-6k^2) (2) - 5 - 15k^2 = 0$

Thus the circle passes through A (05) [10]

If $k = \frac{1}{2}$ then we have

$\frac{3}{4}x^{12} + \frac{3}{4}y^{12} + 6x^1 - \frac{1}{2}y^1 - \frac{35}{4} = 0$

The equation of the circle is $S_3 \equiv x^2 + y^2 + 8x - \frac{2}{3}y - \frac{35}{3} = 0$ (05) [05]

$C_3 = \left(-4, \frac{1}{3}\right)$ and $r_3 = \sqrt{16 + \frac{1}{9} + \frac{35}{3}} = \frac{5\sqrt{10}}{3}$ (05)

$C_2C_3 = \sqrt{8^2 + \left(\frac{8}{3}\right)^2} = \frac{8\sqrt{10}}{3}$ (05)

$= \frac{5\sqrt{10}}{3} + \sqrt{10}$

$= r_3 + r_2$ (05)

Thus the two circles $S_3 = 0$ and $S_2 = 0$ touch each other externally at the point A. [15]

$C_1C_3 = \sqrt{2^2 + \left(\frac{2}{3}\right)^2} = \frac{2\sqrt{10}}{3}$ (05)

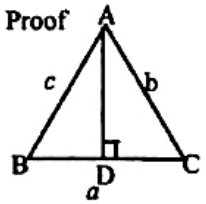
$= \frac{5\sqrt{10}}{3} - \sqrt{10}$

$= r_3 - r_1$ (05)

Thus the two circle $S_1 = 0$ and $S_3 = 0$ touch each internally at the point A. [10]

(09) (a) Cosine rule $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ (05) [05]

Proof



$AB^2 = BD^2 + AD^2$

$= (BC - DC)^2 + (AC^2 - DC^2)$

$= BC^2 + AC^2 - 2BC \cdot DC$

$= BC^2 + AC^2 - 2BC \cdot AC \cos C$

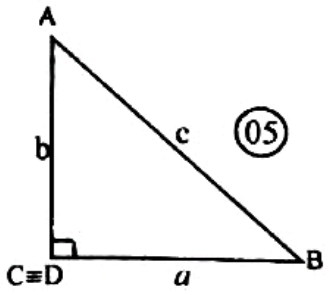
(i). ABC is an acute angled triangle

$C^2 = a^2 + b^2 - 2ab \cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ (05)

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(ii) ABC is an right angled triangle



(05)

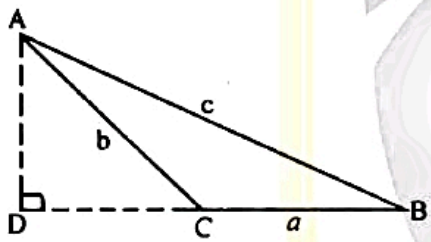
$AB^2 = AC^2 + BC^2$

$= AC^2 + BC^2 - 2AC\,BC\cdot\cos C \quad \therefore C = \frac{\pi}{2}$

$C^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2+b^2-c^2}{2ab}$ (05)

(iii) ABC is an obtuse angled triangle



$AB^2 = AD^2 + DB^2$

$= AC^2 - DC^2 + (DC + BC)^2$

$= AC^2 + BC^2 + 2BC\cdot AC\cos(\pi - C)$

$= AC^2 + BC^2 - 2BC\cdot AC\cos C$ (05)

$c^2 = b^2 + a^2 - 2ab\cos C$

$\cos C = \frac{a^2+b^2-c^2}{2ab}$ [20]

(i) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$ (05)

$2\left(\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{C}\right) = \frac{a^2+b^2-c^2}{abc}$ (05)

[10]

(ii) $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

$\frac{a+b+2c}{(a+c)(b+c)} = \frac{3}{a+b+c}$

$(a+b+2c)(a+b+c) = 3(a+c)(b+c)$ (05)

$(a+b)^2 + 3c(a+b) + 2c^2 = 3\left(ab + bc + ac + c^2\right)$

$a^2 + b^2 - c^2 = ab$ (05)

$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{1}{2} \Rightarrow c = \frac{\pi}{3}$ (05) [15]

(b) $\sqrt{3}\cos\theta + \sin\theta = 2\left\{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right\}$

$= 2\left(\cos\frac{\pi}{6}\cos\theta + \sin\frac{\pi}{6}\sin\theta\right)$

$= 2\cos\left(\theta - \frac{\pi}{6}\right)$ (05)

$R\cos(\theta - \alpha)$. *Where* $R = 2$ *and* $\alpha = \frac{\pi}{6}$ (05)

[10]

$\sqrt{3}\cos^2\theta + \left(1-\sqrt{3}\right)\sin\theta\cos\theta - \sin^2\theta = \cos\theta - \sin\theta$

$\left(\sqrt{3}\cos\theta + \sin\theta\right)(\cos\theta - \sin\theta) = \cos\theta - \sin\theta$ (05)

$\left(\sqrt{3}\cos\theta + \sin\theta - 1\right)(\cos\theta - \sin\theta) = 0$

$\left(\sqrt{3}\cos\theta + \sin\theta\right) = 1$ *or* $\cos\theta - \sin\theta = 0$ (05)

$\cos\theta - \sin\theta = 0 \Rightarrow \tan\theta = 1$

Thus the general solution is $\theta = n\pi + \frac{\pi}{4}$, Where n is an integer. (05)

$\sqrt{3}\cos\theta + \sin\theta = 1 \Rightarrow 2\cos\left(\theta - \frac{\pi}{6}\right) = 1$

$\cos\left(\theta - \frac{\pi}{6}\right) = \frac{1}{2} = \cos\frac{\pi}{3}$

$\theta = 2m\pi \pm \frac{\pi}{3} + \frac{\pi}{6}$, *where m is an integer.* (05) [20]

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(c) Case (1) suppose that x is positive.

Let $\theta = \cos^{-1} x$

Then θ lies between O and $\frac{\pi}{2}$ (05)

$x = \cos\theta$

Thus we have $-x = -\cos\theta = \cos(\pi - \theta)$

This gives $\cos^{-1}(-x) = \pi - \theta = \pi - \cos^{-1} x$ (05)

case (2) Suppose that x is negative.

write $x = -y$, where y is positive.

By case (1) we have $\cos^{-1}(-y) = \pi - \theta = \pi - \cos^{-1} y$

Writing $x = -y$ in above equation we have

$\cos^{-1} x = \pi - \theta = \pi - \cos^{-1}(-x)$

i. e. $\cos^{-1} x = \pi - \cos^{-1}(-x)$ (05)

case (3) suppose that $x = 0$

Then $\cos^{-1} x = \cos^{-1} 0 = \frac{\pi}{2}$ and $\cos^{-1}(-x) = \cos^{-1}(0) = \frac{\pi}{2}$

Which implies $\cos^{-1} x = \pi = \cos^{-1}(-x)$ (05) [20]