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G.C.E. (Advanced Level) Examination - August 2009
10 - Combined Mathematics - I
Marking Scheme

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(01) Since α and β are the roots of the equation

$x^2 + bx + c = 0$

$\alpha + \beta = -b$ and $\alpha\beta = c$ $\rightarrow$(01) (05)

We have $\alpha^3\beta^2 + \alpha^2\beta^3 = \alpha^2\beta^2(\alpha + \beta)$ and

$(\alpha^3\beta^2) + (\alpha^2\beta^3) = (\alpha\beta)^5 \rightarrow$(02) (05)

From (01) and (02) we have

$\alpha^3\beta^2 + \alpha^2\beta^3 = -bc^2$ and $(\alpha^3\beta^2)\ (\alpha^2\beta^3) = c^5$ (05)

Thus $x^2 + bc^2x + c^5 = 0$ $\rightarrow$(03) is the equation whose roots are $\alpha^3\beta^2$ and $\alpha^2\beta^3$ (05) [20]

$\alpha^3\beta^2 + \frac{1}{\alpha^2\beta^3} = \frac{\alpha^5\beta^5 + 1}{\alpha^2\beta^3} = \frac{c^5 + 1}{\alpha^2\beta^3}$ *from* (01) (05)

similarly, $\alpha^2\beta^3 + \frac{1}{\alpha^3\beta^2} = \frac{\alpha^5\beta^5 + 1}{\alpha^3\beta^2} = \frac{c^5 + 1}{\alpha^3\beta^2}$ (05)

write $y = \frac{c^5 + 1}{x}$ (10)

Substituting for x in the equation (03) we have

$$\left(\frac{c^5 + 1}{y}\right)^2 + bc^2\frac{(c^5 + 1)}{y} + c^5 = 0$$ (05)

Hence, the equation whose roots are $\alpha^3\beta^2 + \frac{1}{\alpha^2\beta^3}$ and $\alpha^2\beta^3 + \frac{1}{\alpha^3\beta^2}$ is given by

$c^5y^2 + bc^2(c^5 + 1)y + (c^5 + 1)^2 = 0$ (05) [30]

(b) When the polynomial $f(x)$ is divided by $(x - \alpha)$, we get $f(x) \equiv (x - \alpha)Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder.

Thus we get $R = f(\alpha)$ (05) [05]

It is given that when the polynomial $f(x)$ is divided by $(x - \alpha)(x - \beta)(x - \gamma)$, where α, β and γ are unequal real numbers, the remainder takes the form

$A(x - \beta)(x - \gamma) + B(x - \alpha)(x - \gamma) + C(x - \alpha)(x - \beta)$

using the above rusult, we have

$f(\alpha) = A(\alpha - \beta)(\alpha - \gamma)$, (05) $f(\beta) = B(\beta - \alpha)(\beta - \gamma)$ and (05)

$f(\gamma) = C(\gamma - \alpha)(\gamma - \beta)$ (05)

Thus

$$A = \frac{f(\alpha)}{(\alpha - \beta)(\alpha - \gamma)}, \quad B = \frac{f(\beta)}{(\beta - \alpha)(\beta - \gamma)}$$

and $C = \frac{f(\gamma)}{(\gamma - \alpha)(\gamma - \beta)}$

and the remainder takes the form

$$\frac{f(\alpha)}{(\alpha - \beta)(\alpha - \gamma)}(x - \beta)(x - \gamma) + \frac{f(\beta)}{(\beta - \alpha)(\beta - \gamma)}(x - \alpha)(x - \gamma)$$

$$+ \frac{f(\gamma)}{(\gamma - \alpha)(\gamma - \beta)}(x - \alpha)(x - \beta)$$ (05) [20]

The coefficient of the term in x is equal to

$$= -\left\{\frac{f(\alpha)}{(\alpha - \beta)(\alpha - \gamma)}(\beta + \gamma) + \frac{f(\beta)}{(\beta - \alpha)(\beta - \gamma)}(\alpha + \gamma) + \frac{f(\gamma)}{(\gamma - \alpha)(\gamma - \beta)}(\alpha + \beta)\right\}$$ (05)

Let $f(x) = x^5 - kx$ and let $\alpha = -1$, $\beta = 1$ and $\gamma = 2$

Then the coefficient of the term in x in the remainder is equal to

$$= -\left\{\frac{f(-1)}{(-1-1)(-1-2)}(1+2) + \frac{f(1)}{(1+1)(1-2)}(-1+2)\right\}$$ (10)

$$= -\left\{\frac{(-1)^5 + k}{2} - \frac{(1)^5 + k}{2}\right\} = -k + 1 = 0 \Rightarrow k = 1$$ (05) (05)

[25]

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(02) *(a)* P - 2, H - 2, O - 2, I - 1, L - 1, S - 1, Y - 1

The number of differnt arrangements of the

10 letters $= \frac{10!}{2!\ 2!\ 2!} = 453600$ (05)

(05)

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By considering HISY as one letter the number of arrangements $= \frac{7!}{2!\ 2!} = 1260$ (05)

The 4 letters H, I, S and Y can be arranged in 4! = 24 ways. (05)

Thus the number of arrangements with H, I, S and Y together including the arrangements where the letters I, S and Y are sand - witched between the two letters of

$$H = \frac{7!}{2!\ 2!} \times 4! = 1260 \times 4 = 30240$$ (05)

The number of arrangements with the letters I, S and Y sand witched between the two letters of

$$H = \frac{6!}{2!\ 2!} \times 3! = 180 \times 6 = 1080$$ (05)

Hence the required number of arragements

$= 30240 - n1080 = 29160$ (05)

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Different selections of 5 letters can be done in one of the following ways.

(i) Two pair of equal letters and one different letter.

(ii) One pair of equal letters and three different letters.

(iii) Five different letters

The number of different selections in method

(i) $= {}^3C_2 \times {}^5C_1 = 15$ (05)

The number of different selections in method

(ii) $= {}^3C_1 \times {}^6C_3 = 60$ (05)

The number of different selections in method

(iii) $= {}^7C_5 = 21$

Hence the total number of selections $= 15 + 60 + 21$

$= 96$ (05)

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(b) Since $P_n = n(n+1) \ldots\ldots (n+r-1)$, where n and r are positive integers, we have

$P_{n+1} = (n+1)(n+2) \ldots\ldots (n+r)$ (05)

Thus $nP_{n+1} = n(n+1)(n+2) \ldots\ldots (n+r)$ (05)

$= (n+r)P_n$

$= nP_n + rP_n$

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Hence $P_{n+1} - P_n = r\left(\frac{P_n}{n}\right)$ (05)

Since it is given that $\left(\frac{P_n}{n}\right)$ is divisible by $(r-1)!$

$P_{n+1} - P_n$ is divisible by $r!$ (10)

Write $P_{n+1} - P_n = k_n \cdot r!$ Where k_n is an integer (05)

We have

$P_1 = r!$

$P_2 - P_1 = k_1\, r!$

$P_3 - P_2 = k_2\, r!$ (05)

.....

.....

$P_{n-1} - P_{n-2} = k_{n-2}\, r!$

$P_n - P_{n-1} = k_{n-1}\, r!$ (05)

$P_n - P_1 = r! \sum_{i=1}^{n-1} k_i \Rightarrow P_n = \left\{1 + \sum_{i=1}^{n-1} k_i\right\} r!$ (05)

Hence the product of r consecutive positive integers is divisible by r! (05)

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(03) **(a)** The result $(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$ is valid fer

n = 1, since ${}^1C_0 = 1 = {}^1C_1$ (05)

Let us assume that the result is valid for n = p

Then $(x+y)^p = \sum_{r=0}^{p} {}^pC_r\, x^{p-r} y^r$ (05)

Multiplying both sides by (x + y) we have

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$(x+y)^{p+1} = \sum_{r=0}^{p} {}^{p}C_r\, x^{p-r+1} y^r + \sum_{r=0}^{p} {}^{p}C_r\, x^{p-r} y^{r+1}$ (05)

The coefficient of $x^{p-r+1} y^r$ in the expansion on the R. H. S. is ${}^{p}C_r + {}^{p}C_{r-1}$ (05)

We have ${}^{p}C_r + {}^{p}C_{r-1} = \frac{p!}{(p-r)!\,r!} + \frac{p!}{(p-r+1)!\,(r-1)!}$

$= \frac{p!}{(p-r)!\,(r-1)!}\left\{\frac{1}{r} + \frac{1}{p-r+1}\right\} = \frac{(p+1)!}{(p+1-r)!\,r!} = {}^{p+1}C_r$ (05)

Thus the result is valid for n = p + 1

Hence, by the principle of Mathematical Induction the result is valid for positive integral n. (05) [30]

Substituting x = p and y = q in $(x+y)^n = \sum_{r=0}^{n} {}^{n}C_r\, x^{n-r}\, y_r$

We get

$(p+q)^n = p^n + \sum_{r=1}^{n-1} {}^{n}C_r\, p^{n-r}\, q^r + q^n$ (05)

$(p+q)^n - p^n - q^n = \sum_{r=1}^{n-1} {}^{n}C_r\, p^{n-r}\, q^r$

$= pq \sum_{r=1}^{n-1} {}^{n}C_r\, p^{n-r-1} - q^{r-1}$ (05)

Since each term in Σ sign is a positive integer.

$(p + q)^n - P^n - q^n$ is divisible by pq. (05) [15]

(b) $U_r = \frac{2r+1}{(3r-2)(3r+1)} \cdot \frac{1}{7^r}$

$Let\ \frac{(2r+1)}{(3r-2)(3r+1)} = \frac{A}{3r-2} - \frac{B}{(3r+1)}$ (05)

Then $2r + 1 = A(3r + 1) - B(3r - 2) \Rightarrow 2r + 1 \equiv 3(A - B) = r + A + 2B$

Thus we have $3(A - B) = 2$ and $A + 2B = 1$

(05)

$\Rightarrow A = \frac{7}{9}$ *and* $B = \frac{1}{9}$ (05)

(05)

$U_r = \frac{2r+1}{(3r-2)(3r+1)} \cdot \frac{1}{7^r}$

$= \frac{1}{9}\left\{\frac{1}{(3r-2)} \cdot \frac{1}{7^{r-1}} - \frac{1}{(3r+1)} \cdot \frac{1}{7^r}\right\}$ (05)

$U_r = f(r-1) - f(r)$, *Where* $f(r) = \frac{1}{9(3r+1)} \cdot \frac{1}{7^r}$ (05) [30]

We have $S_n = \sum_{r=1}^{n}\{f(r-1) - f(r)\}$ (05)

$= f(0) - f(n) = \frac{1}{9} - \frac{1}{9(3n+1)} \cdot \frac{1}{7^n}$

(05) (05) [15]

Further $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left\{\frac{1}{9} - \frac{1}{9(3n+1)} \cdot \frac{1}{7^n}\right\} = \frac{1}{9}$ (05)

$\therefore \lim_{n\to\infty} \frac{1}{9(3n+1)} \cdot \frac{1}{7^n} = 0$ (05) [10]

(04) (a) Let $a + ib$ be a square root of -80 - 18i, where a and b are real.

Then $(a + ib)^2 = -80 - 18i$

$\Rightarrow a^2 - b^2 + 12ab = -80 - 18i$ (05)

Equating the real and imaginary parts, We get

$a^2 - b^2 = -80$ and $2ab = -18$ (05)

using $b = -\frac{9}{a}$ in $a^2 - b^2 = -80$ we get

$a^2 - \frac{81}{a^2} = -80 \Rightarrow a^4 + 80a^2 - 81 = 0$

$\Rightarrow (a^2 - 1)(a^2 + 81) = 0 \Rightarrow a^2 = 1\ or\ -81$

But $a^2 \neq -81$, $\therefore$ a is real. (05)

Thus $a = 1$ or -1, and $b = -9$ or 9

Hence the square roots of -80 - 18i are

1 - 9i and -1 + 9i (05) [20]

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Discriminant of the quadratic equation

$4z^2 + (16i - 4)z + (65 + 10i) = 0$ is

$(16i - 4)^2 - 16(65 + 10i) = (-256 - 128i + 16) - (1040 + 160i)$

$= (-1280 - 288i) = 16(-80 - 18i)$ (05)

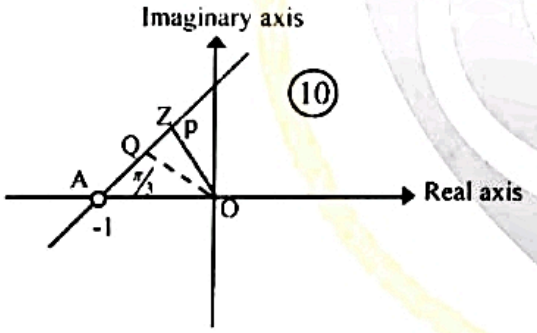
Since $-80 - 18i = (1 - 9i)^2$ the discriminant is equal to $\{4(1 - 9i)\}^2$ (05)

Thus the roots of the quadratic equation are

$$\frac{-(16i-4)\pm 4(1-9i)}{8} = \frac{8-52i}{8} \quad and \quad \frac{20i}{8}$$

The roots are $1-\frac{13}{2}i$ (05) *and* $\frac{5}{2}i$ (05) [20]

(b) The equation $\arg(z+1) = \frac{\pi}{3}$ represents the line segment starting from point A, but excluding A, and making an angle $\frac{\pi}{3}$ with the real axis.



(10)

$|z| = OP$ and $|z|$ is minimum

When OP is perpendicular to the line. (05)

Thus the minimum value of $|z| = OQ = \frac{\sqrt{3}}{2}$ (05) [20]

(c) We have $z^3 - 1 = (z-1)(z^2+z+1)$

Thus the complex roots of $z^3 - 1 = 0$ are the roots of the equation $z^2 + z + 1 = 0$ (05)

Hence the complex roots of $z^3 - 1 = 0$ are $\frac{-1\pm\sqrt{3}i}{2}$ (05)

Let $\omega = \frac{-1+\sqrt{3}i}{2}$ *Then* $\omega^2 = \frac{1-3-2\sqrt{3}i}{4} = \frac{-1-\sqrt{3}i}{2}$

Also, $\left(\frac{-1-\sqrt{3}i}{2}\right)^2 = \frac{1-3+2\sqrt{3}i}{4} = \frac{-1+\sqrt{3}i}{2}$

Thus if ω is a complex root of $z^3 - 1 = 0$,

Then the other root is ω^2 (05) [15]

$\omega^2 + \omega + 1 = 0 \Rightarrow \omega + 1 = -\omega^2$

Thus we have $(\omega + 1)^k = (-\omega^2)^k$, where k is an integer.

If k is an odd positive integer, then $(-1)^k = -1$

Thus, if k is an odd positive integer, then

$(\omega + 1)^k + \omega^{2k} = 0$ (05) [05]

Let $f(x) = (x + 1)^n + x^{2n}$, where n is an odd positive integer.

Then $f(\omega) = 0$ (05)

we have

$f(\omega^2) = (\omega^2)^{2n} + (1 + \omega^2)^n = \omega^{4n} + (-\omega)^n = (\omega^3)^n\ \omega^n - \omega^n = 0$

Thus $(x - \omega)$ and $(x - \omega^2)$ are factors of $f(x)$ (05)

further the product $(x - \omega)(x - \omega^2) = x^2 - (\omega + \omega^2)x + \omega^3$

$= x^2 + x + 1$ (05)

Hence $x^2 + x + 1$ is a factor of $x^{2n} + (1 + x)^n = 0$ (05) [20]

(05) (a)

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{2\cos\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$ (05)

$$= \lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\cos\left(x+\frac{\Delta x}{2}\right)$$ (05)

$= 1 \cdot \cos x \quad \therefore \lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)} = 1$ (05)

$= \cos x$ [15]

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$g(x) = \cos x = \sin\left(\frac{\pi}{2} - x\right) = f\left(\frac{\pi}{2} - x\right) = f(y).$

Where $y = \frac{\pi}{2} - x \quad \therefore dy = -dx$

$\frac{dg(x)}{dx} = \frac{df(y)}{dx} = \frac{df(y)}{dy} \cdot \frac{dy}{dx} = \cos y.(-1)$ (05)

$= -\cos\left(\frac{\pi}{2} - x\right)$

$= -\sin x$ (05) [10]

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We have $\sin^2 x + \cos^2 = 1$

Differentiating the above equation with respect to x we get

$2\sin x \cos x + 2\cos x \frac{d\cos x}{dx} = 0$ (05)

Thus $\frac{d\cos x}{dx} = -\sin x$ (05) [10]

(i) $\frac{d}{dx}\sin\left[\ell n\left(1+x^2\right)\right] = \cos\left[\ell n\left(1+x^2\right)\right] \cdot \frac{1}{\left(1+x^2\right)} \cdot 2x$

(05) (05)

$= \frac{2x}{\left(1+x^2\right)} = \cos\left[\ell n\left(1+x^2\right)\right]$ (05) [15]

(ii) $\frac{d}{dx}\cos(\sin x) = -\sin(\sin x) \cdot \cos x$ (05)

(05)

$= -\cos x \sin(\sin x)$ [10]

(b) (i)

$y = \sin k\theta \operatorname{cosec}\theta$ and $x = \cos\theta$ where k is a constant.

$\frac{dy}{d\theta} = k\cos k\theta \cdot \operatorname{cosec}\theta - \sin k\theta \times \cot\theta \operatorname{cosec}\theta$

and $\frac{dx}{d\theta} = -\sin\theta$ (05)

$\frac{dy}{dx} = -k\cos k\theta \operatorname{cosec}^2\theta + \sin k\theta \cot\theta \operatorname{cosec}^2\theta$ (05)

$\sin^2\theta \frac{dy}{dx} = -k\cos k\theta + \sin k\theta \cot\theta$

$\left(1-x^2\right)\frac{dy}{dx} - xy + k\cos k\theta = 0$ (05) [15]

(ii) Differentiating above equation with respect to x we have

$\left(1-x^2\right) = \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - y - x\frac{dy}{dx} + k\frac{d\cos k\theta}{d\theta} \cdot \frac{d\theta}{dx} = 0$ (05)

$\left(1-x^2\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y - k^2\sin k\theta \cdot \left(-\frac{1}{\sin\theta}\right) = 0$ (05)

$\left(1-x^2\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + \left(k^2-1\right)y = 0$ (05) [15]

(c) If $y\left(1+x^2\right) = 2, \left(1+x^2\right)\frac{dy}{dx} + 2xy = 0$ (05)

At $P\left(3, \frac{1}{5}\right), \frac{dy}{dx} = \frac{-3}{25}$ and so the equation of the tangent at P is,

$= \frac{y - \frac{1}{5}}{x-3} = \frac{-3}{25} \Rightarrow 3x + 25y = 14$ (05)

This tangent meets the curve again when $\frac{2}{1+x^2} = \frac{14-3x}{25}$

i.e. when $3x^3 - 14x^2 + 3x + 36 = 0$ (05)

This equation gives the abcissue of the three points in which the tangent at P meets the curve and since two of these points coints coincide at P we know that $(x - 3)^3$ is a factor of $3x^3 - 14x^2 + 3x + 36$,

The remaining factor is $(3x + 4)$ when ce $Q = \left(-\frac{4}{3}, \frac{18}{25}\right)$ (05)

[20]

(06) (a) Let $I_k = \int \frac{e^t}{t^k} dt$, where $t > 0$ and k is a positive integer.

Then $I_k = \int \frac{e^t}{t^k} dt = \int \frac{(-1)}{(k-1)} e^t \frac{d}{dt}\left(\frac{1}{t^{k-1}}\right) dt \quad k \neq 1$ (10)

$= -\frac{1}{k-1}\frac{e^t}{t^{k-1}} + \frac{1}{k-1}\int \frac{e^t}{t^{k-1}} dt$

Thus $I_k - I_{k-1} + \frac{1}{k-1} \cdot \frac{e^t}{t^{k-1}} = constans \rightarrow (01)$

(10) [20]

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Let $I = \int e^x \left(\frac{1-x}{1+x}\right)^2 dx$, and let $t = 1 + x$ (05)

Then $I = \int e^{t-1} \frac{(2-t)^2}{t^2} dt = \frac{1}{e} \int e^t \left\{\frac{4}{t^2} - \frac{4}{t} + 1\right\} dt$ (10)

Since $I_k = \int \frac{e^t}{t^k} dt$, we have $I = \frac{1}{e}\left\{4I_2 - 4I_1 + \int e^t dt\right\}$ (05)

From (1), we have $I_2 - I_1 = constant - \frac{e^t}{t}$ (05)

Thus $I = \frac{1}{e}\left\{constant - \frac{4e^t}{t} + \int e^t dt\right\}$

$= \frac{1}{e}\left\{constant - \frac{4e^t}{t} + e^t\right\}$ (05)

$= \frac{e^{1+x}}{e} - \frac{4}{e}\frac{e^{1+x}}{(1+x)} + constant$ 30

(b) Let $J = \int_0^a f(x)\,dx$, where $a > 0$

Let $y = a - x$. Then $dy = -dx$ and (05)

$J = -\int_a^0 f(a-x)\,dx = \int_0^a f(a-x)\,dx \rightarrow (01)$ (05) 20

From (01), we have

$\int_0^{\pi/2} \sin^{2k} x\,dx = \int_0^{\pi/2} \sin^{2k}\left(\frac{\pi}{2} - x\right)dx = \int_0^{\pi/2} \cos^{2k} x\,dx$ (10)

Where k is a positive integer.

Thus

$\int_0^{\pi/2} \frac{\sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx = \int_0^{\pi/2} \frac{\cos^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx$ (10)

$\Rightarrow \int_0^{\pi/2} \frac{\sin^{2k} x}{\sin^{2k} x + \cos^{2k} x} dx$

$= \frac{1}{2}\int_0^{\pi/2} dx = \frac{\pi}{4}$ (10) 30

(07) The equation of a line perpendicular to the line $ax + by + c = 0$ takes the form $bx - ay - c^1 = 0$ (05)

Thus the equation of the line passing through the point $(x_0\ y_0)$ and perpendicular to the line $ax + by + c = 0$, is $bx - ay - bx_0 - ay_0 = 0$ (05)

We have $b(x - x_0) - a(y - y_0) = 0$

Thus $\frac{(x - x_0)}{a} = \frac{y - y_0}{b} = t\ (say)$ (05)

Hence, the cordinates of any point on the line passing through the point $(x_0\ y_0)$ and perpendicular to the line $ax + by + c = 0$ can be expressed in the form $(x_0 + at,\ y_0 + bt)$ where is a parameter. 15

Let $(x_0 + at,\ y_0 + bt)$ be the mirror image of the point $(x_0\ y_0)$ in the straight line $ax + by + c = 0$

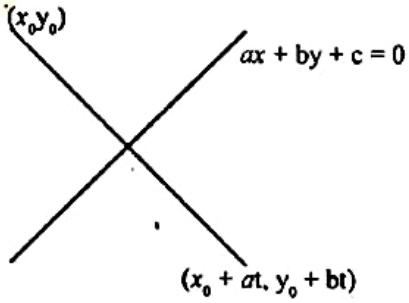
Since $\left[\frac{2x_0 + at}{2}, \frac{2y_0 + bt}{2}\right]$ the mid point of $(x_0\ y_0)$ and $(x_0 + at,\ y_0 + bt)$ lies on the line $ax + by + c = 0$ we have

$a\frac{2x_0 + at}{2} + b\frac{2y_0 + bt}{2} + c = 0$ (05)

Thus, we get $t = -\frac{2(ax_0 + by_0 + C)}{a^2 + b^2}$ (05)

The mirror image of the point $(x_0\ y_0)$ in the straight line $ax + by + c = 0$ is

$(x_0 + at,\ y_0 + bt)$ where $t = -\frac{2(ax_0 + by_0 + C)}{a^2 + b^2}$



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The equation of the perpendicular bisecter of the side OA is $x\cos\theta + y\sin\theta - 1 = 0$

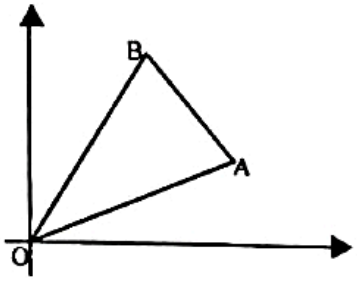
Since the point A is the mirror image of O in the line $x\cos\theta + y\sin\theta - 1 = 0$ the coordinates of A are $(2\cos\theta, 2\sin\theta)$

(05) (05)

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The equation of the side OA is

$\frac{y-0}{x-0}=\frac{2\sin\theta-0}{2\cos\theta-0}$ (05)



i.e. $y = x\tan\theta$

The equation of the perpendicular bisector of the side AB is

$x - y - 1 = 0$

Since the point B is the mirror image of A in the line $x - y - 1 = 0$, the coordinates of B are

$(1 + 2\sin\theta , -1 + 2\cos\theta)$

(05) (05)

The equation of the side OB is $\frac{y-0}{x-0}=\frac{-1+2\cos\theta-0}{1+2\sin\theta-0}$ (05)

i.e. $x(2\cos\theta - 1) - y(2\sin\theta + 1) = 0$

The equation of the side AB is

$\frac{y-2\sin\theta}{x-2\cos\theta}=\frac{-1+2\cos\theta-2\sin\theta}{1+2\sin\theta-2\cos\theta}$ (10) 40

The equation of the perpendicular bisector of the side OB is

$y-\left(\cos\theta-\frac{1}{2}\right)=\frac{-(1+2\sin\theta)}{(1-2\cos\theta)}\left\{x-\left(\sin\theta+\frac{1}{2}\right)\right\}$ (10)

$y(2\cos\theta-1)+x(1+2\sin\theta)-\frac{1}{2}\left\{(2\cos\theta-1)^2+(2\sin\theta+1)^2\right\}=0$

$y(2\cos\theta-1)+x(1+2\sin\theta)-(3-2\cos\theta+2\sin\theta)=0$ (05)

15

The coordinates of the point of intersection of the two perpendicular bisectors of the sides OA and AB are given by the two equation $x\cos\theta + y\sin\theta - 1 = 0$ and $x - y - 1 = 0$

solving the above two equations, we get

$x=\frac{1+\sin\theta}{\cos\theta+\sin\theta}$ and $y=\frac{1-\cos\theta}{\cos\theta+\sin\theta}$

(05) (05)

The coordinates of the point of intersection of the two perpendicular bisectors of the sides OA and AB are

$\left(\frac{1+\sin\theta}{\cos\theta+\sin\theta}, \frac{1-\cos\theta}{\cos\theta+\sin\theta}\right)$

Since

$\frac{(1-\cos\theta)(2\cos\theta-1)}{\cos\theta+\sin\theta}+\frac{(1+\sin\theta)(1+2\sin\theta)}{\cos\theta+\sin\theta}-(3-2\cos\theta+2\sin\theta)$ (05)

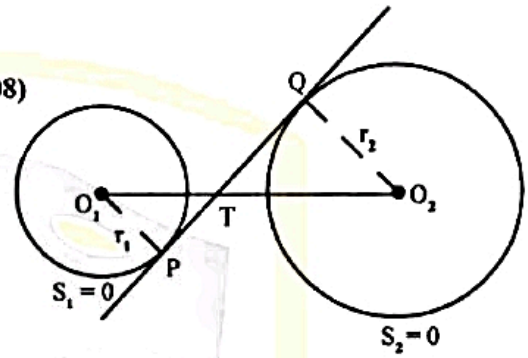
$=\frac{\left(-1+3\cos\theta-2\cos^2\theta\right)+\left(1+3\sin\theta+2\sin^2\theta\right)}{\cos\theta+\sin\theta}-(3-2\cos\theta+2\sin\theta)$

$=\frac{3(\cos\theta+\sin\theta)+2\left(\sin^2\theta-\cos^2\theta\right)}{\cos\theta+\sin\theta}-(3-2\cos\theta+2\sin\theta)=0$ (05)

Thus the point of intersection of the two perpendicular bisectors of the sides OA and AB lies on the perpendicular bisector of the side OB

i.e. the three perpendicular bisectors are concurrent. 20

(08)



Let PQ be a common tangent to the circle

$S_1 \equiv x_1 + y_1 + 2g_1x + 2f_1y + c_1 = 0$ and

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Let PQ intersect O_1O_2 between O_1 and O_2 Then O_1 P and O_2Q are perpendicular to PQ Let T be the point of intersection of PQ and O_1O_2

Since the triangles O_1 PT and O_2 QT are similar,

$\frac{O_1T}{TO_2}=\frac{r_1}{r_2}$, where r_1 and r_2 are the radius of the two circles

(10)

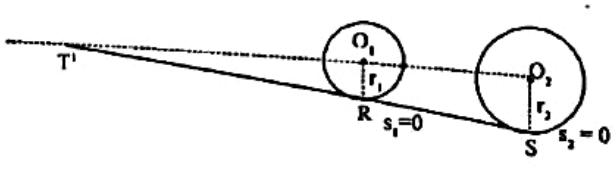
$S_1 = 0$ and $S_2 = 0$ respectively.

Thus T is the point that divides O_1O_2 in ternally in th ratio $r_1 : r_2$ and (05)

$T \equiv \left[\frac{-g_1r_2-g_2r_1}{r_1+r_2}, \frac{-f_1r_2-f_2r_1}{r_1+r_2}\right]$

(05) (05)

Clearly, another tangent can be drawn to the two circles from T. 25



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Let the common tangent RS to the two circles $S_1 = 0$ and $S_2 = 0$ meet O_1O_2 at T^1

Since the triangles O_1T^1R and O_2T^1S are similar,

we get $\frac{O_1T^1}{O_2T^1} = \frac{r_1}{r_2}$ (10)

Thus T^1 is the point that divides O_1O_2 externally in the (05)

ratio $r_1 : r_2$ and $T^1 \equiv \left[\frac{g_1r_2 - g_2r_1}{r_1 - r_2}, \frac{-f_1r_2 - f_2r_1}{r_1 - r_2}\right]$ (05) (05)

Clearly, another tangent can be drawn to the two circles from T^1. 25

Equations, of the two circles can be written as

$(x - 9)^2 + (y + 3)^2 = 2^2$ and $(x + 9)^2 + (y - 3)^2 = 4^2$

Thus the coordinates T and T^1 are

$\left[\frac{9\times4 - 9\times2}{2+4}, \frac{-3\times4 + 3\times2}{2+4}\right] = (3, -1)$ and (05)

$\left[\frac{-2\times9 + 4\times(-9)}{2-4}, \frac{-2\times(-3) + 4\times3}{2-4}\right] = (27, -9)$ (05)

respectively.

Equation of a line ℓ through T takes the form $y + 1 = m(x - 3)$

i.e. $y - mx + 3m + 1 = 0$ (05)

Either

If the line ℓ is a tangent to the circle $(x - 9)^2 + (y + 3)^2 = 2^2$

Then, $\left|\frac{-3 - 9m + 3m + 1}{\sqrt{1 + m^2}}\right| = 2$ (05)

i.e. $(-6m - 2)^2 = 4(1 + m^2) \Rightarrow 8m^2 + 6m = 0$

$\Rightarrow m = 0$ or $-\frac{3}{4}$ (05)

or

If the line ℓ is a tangent to the circle $(x + 9)^2 + (y - 3)^2 = 4^2$

Then $\left|\frac{3 + 9m + 3m + 1}{\sqrt{1 + m^2}}\right| = 4$ (05)

i.e.

$4(3m + 1) = 4\sqrt{1 + m^2} \Rightarrow (3m + 1)^2 = 1 + m^2 \Rightarrow 8m^2 + 6m = 0$

$\Rightarrow m = 0$ or $-\frac{3}{4}$ (05)

Thus the equation of the two common tangents from T to the two circles are $y + 1 = 0$ and $y + 1 = -\frac{3}{4}(x - 3)$ (05)

i.e. $y + 1 = 0$ and $4y + 3x - 5 = 0$

Equation of a line ℓ through T^1 takes the form

$y + 9 = m^1(x - 27)$ (05)

i.e. $y - m^1x + 27m + 9 = 0$

Either

If the line ℓ^1 is a tangent to the circle $(x - 9)^2 + (y + 3)^2 = 2^2$

Then $\left|\frac{-3 - 9m^1 + 27m^1 + 9}{\sqrt{1 + m^2}}\right| = 2$ (05)

i.e. $(18m^1 + 6)^2 = 4(1 + m^{12}) \Rightarrow 80m^{12} + 54m^1 + 8 = 0$

$\Rightarrow 40m^{12} + 27m^1 + 4 = 0$

$m^1 = \frac{-27 \pm \sqrt{27^2 - 640}}{80} = \frac{-27 \pm \sqrt{729 - 640}}{80} = \frac{-27 \pm \sqrt{89}}{80}$ (05)

or

If the line ℓ^1 is a tangent to the circle

$(x + 9)^2 + (y - 3)^2 = 4^2$ Then

$\left|\frac{3 + 9m^1 + 27m^1 + 9}{\sqrt{1 + m^{12}}}\right| = 4$ (05)

i.e. $(9m^1 + 3)^2 = (1 + m^{12}) \Rightarrow 80m^{12} + 54m^1 + 8 = 0$

$\Rightarrow 40m^{12} + 27m^1 + 4 = 0$

$m^1 = \frac{-27 \pm \sqrt{27^2 - 640}}{80} = \frac{-27 \pm \sqrt{729 - 640}}{80} = \frac{-27 \pm \sqrt{89}}{80}$ (05)

Thus the equation of the two common tangents from T^1 to the two circles are

$y + 9 = \frac{-27 + \sqrt{89}}{80}(x - 27)$ and $y + 9 = \frac{-27 - \sqrt{89}}{80}(x - 27)$ (05)

70

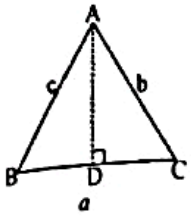
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(09) (a) sine rule in the usual notation

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ (05) [05]

proof :

(i) ABC is an acute angled triangle

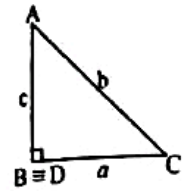


AD = AB sin B = AC sin C

$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$ (05)

similarly $\frac{\sin A}{a} = \frac{\sin C}{b}$

(ii) ABC is an right angled triangle

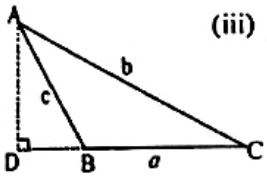


AD = AB sin B = AC sin C

$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$ (05)

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{b}$ (05)

(iii) ABC is an obtuse angled triangle.



AD = AB sin (π – B) = AC sin C

$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$ (05) [20]

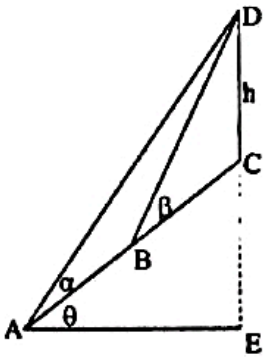
(i) applying sine rule to triangle ACD we have

$$\frac{AD}{\sin\left(\frac{\pi}{2}+\theta\right)} = \frac{h}{\sin\alpha} \Rightarrow h = \frac{AD\sin\alpha}{\cos\theta}$$ (05)

applying sine rule to trianle ABD we have

$$\frac{AD}{\sin(\pi-\beta)} = \frac{x}{\sin(\beta-\alpha)} \Rightarrow AD = \frac{x\sin\beta}{\sin(\beta-\alpha)}$$ (05)

$$\therefore h = \frac{x\sin\beta\sin\alpha}{\sin(\beta-\alpha)\cos\theta}$$ (05)



(05) [20]

(ii) d = DE = AD sin (α + θ) (05)

$$AD = \frac{x\sin\beta}{\sin(\beta-\alpha)}$$

$$d = \frac{x\sin\beta\sin(\alpha+\theta)}{\sin(\beta-\alpha)}$$ (05) [10]

(b) (i) The equation sinθ - cosθ = 1 can be expressed in the form $\sin\left(\theta-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ (10)

The general solution of the above equation is

$$\theta - \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4}$$

i.e. $\theta = n\pi + \left\{1+(-1)^n\right\}\pi/4$, where n is any integer. (10)

Note that when n = 0, $\theta = \frac{\pi}{2}$ and when n = 1, $\theta = \pi$ [20]

(ii) $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3} = \sin^{-1}x$

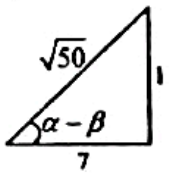
Let $\alpha = \tan^{-1}\frac{1}{2}$ and $\beta = \tan^{-1}\frac{1}{3}$ (05)

then $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{1}{2}-\frac{1}{3}}{1+\frac{1}{2}\times\frac{1}{3}} = \frac{1}{7}$$ (05)

(05)

$\sin(\alpha-\beta) = \frac{1}{\sqrt{50}}$ (05)



Then $\alpha - \beta = \frac{1}{\sqrt{50}}\sin^{-1}\left(\frac{1}{\sqrt{50}}\right)$

But $\alpha - \beta = \sin^{-1}(x)$

$x = \frac{1}{\sqrt{50}} = \frac{\sqrt{2}}{10}$ (05) [25]