G.C.E. (Advanced Level) Examination - August 2008
10 - Combined Mathematics - I
Marking Scheme

(01) *(a)* $\alpha + \beta = -b \quad \alpha\beta = C$ (05)

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = b^2 - 2C$ (05)

$\alpha^4 + \beta^4 = (\alpha + \beta)^4 - 4\alpha\beta(\alpha + \beta)^2 - 6(\alpha\beta)^2$

$= b^4 - 4C(b^2 - 2C) - 6C^2$

$= b^4 - 4cb^2 + 2C^2 \rightarrow (01)$

$\alpha^4 \beta^4 = (\alpha\beta)^4 = C^4$ (05)

Hence the quadratic equation whose roots are α^4 and β^4 given by,

$x^2 - (b^4 - 4cb^2 + 2C^2)x + C^4 = 0 \rightarrow (02)$ (05) [20]

Consider $\frac{\alpha^4}{\beta^4} + 1$

$= \frac{\alpha^4 + \beta^4}{\beta^4}$

$= \frac{b^4 - 4cb^2 + 2c^2}{\beta^4}$ (By usin g(1)) (05)

Similarly $\frac{\beta^4}{\alpha^4} + 1 = \frac{b^4 - 4cb^2 + 2c^2}{\alpha^4}$ (05)

There fore, put $x = \frac{b^4 - 4cb^2 + 2c^2}{y}$ (10)

in the euqation (2) to obtain the required quadratic equation;

$\left(\frac{b^4 - 4cb^2 + 2c^2}{y}\right)^2 - \frac{\left(b^4 - 4cb^2 + 2c^2\right)^2}{y} + c^4 = 0$ (05)

$\Rightarrow c^4 y^2 - \left(b^4 - 4cb^2 + 2c^2\right)^2 y + \left(b^4 - 4cb^2 + 2c^2\right)^2 = 0 \rightarrow (3)$ (05)

[30]

(b) $f(x) \equiv (x - \alpha)Q(x) + R$ (05)

$\rightarrow f(\alpha) = 0.Q(a) + R = R = (= \text{Remainder})$ [05]

$f(x) = (x - \alpha)(x - \beta)S(x) + (Ax + B); \quad \beta \neq \alpha$ (05)

$\Rightarrow f(\alpha) = A\alpha + B;$ (05)

$\Rightarrow f(\beta) = A\beta + B;$

$\Rightarrow A = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$ and $B = \frac{\beta f(\alpha) \alpha f(\beta)}{\beta - \alpha}$

$= \text{constant term}$ (10) [20]

Let $f(x) : x^3 + kx + k$

set $B = 0$ (05)

Take $\alpha = 1, \beta = -2$ or $\alpha = -2, \beta = 1$ (10)

Then $0 = \frac{[2f(1) - f(-2)]}{-3}$

$\Rightarrow f(-2) = -2f(1)$ (05)

$\Rightarrow -8 + 4k + k = -2[1 + k + k]$

$\Rightarrow -8 + 5k = -2 - 4k$

$\Rightarrow 9k = 6$

$\Rightarrow k = \frac{2}{3}$ (05) [25]

(02) *(a)*

(i) Number of ways of selecting 2 girls from among 7 girls $= {}^7C_2$

Number of ways of selecting 3 boys from among 8 boys $= {}^8C_3$ (05)

$\therefore$ The required number of ways $= {}^7C_2 \cdot {}^8C_3$ (05)

$= \frac{7!}{2!\ 5!} \cdot \frac{8!}{3!\ 5!}$

$= 21 \times 56$

$= 1176$ (05) [15]

(ii) "At most 3 boys" occur in one and only one of the fol lowing ways.

No boys and 5 girls

1 boy and 4 girls

2 boys and 3 girls

3 boys and 2 girls

∴ The required number of ways

$= {}^8C_0 \times {}^7C_5 + {}^8C_1 \times {}^7C_4 + {}^8C_2 \times {}^7C_3 + {}^8C_3 \times {}^7C_2$ (15)

$= (1 \times 21) + (8 \times 35) + (28 \times 35) + (56 \times 21)$

$= 21 + 1260 + 1176$

$= 2457$ (05) 20

Aliter.

"More than 3 boys" occur in one and only one of the following ways

4 boys and 1 girl

5 boys and no girls

∴ The number of ways of selecting a team with more than 3 boys

$= {}^8C_4 \times {}^7C_1 + {}^8C_5 \times {}^7C_0$ (05)

$= (70 \times 7) + (56 \times 1)$

$= 546$ (05)

Total number of 5 member teams $= {}^{15}C_5$

$= 3003$ (05)

The required number of ways $= 3003 - 546$

$= 2457$ (05) 20

(iii) Let B_1 and G_1 be the boy and the girl respective that cannot be selected to the same team.

Total number of 5 member teams $= {}^{15}C_5$

The number of teams that can be selected with B_1 and G_1 both in the team $= {}^{13}C_3$ (05)

The required number of ways $= {}^{15}C_5 - {}^{13}C_3$ (05)

$= \frac{15!}{5!\ 10!} - \frac{13!}{3!\ 10!}$

$= 13 \times 11 \times 19$

$= 2717$ (05) 15

Aliter

The number of teams that can be selected with B_1, but without $G_1 = {}^{13}C_4$

The number of teams that can be selected with G_1, but without $B_1 = {}^{13}C_4$

The number of teams that can be selected without both B_1 and $G_1 = {}^{13}C_5$ (05)

∴ The required number of ways $= 2 \times {}^{13}C_4 - {}^{13}C_5$ (05)

$= 2717$ (05) 15

(b) Let the first of the three consecutive coefficients be ${}^nC_r$

Then the second and the third are ${}^nC_{r+1}$ are ${}^nC_{r+2}$ respectively. (05)

It is given that ${}^nC_r = 45$, ${}^nC_{r+1} = 120$ and ${}^nC_{r+2} = 210$

Since $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r+1)!(n-r-1)!}} = \frac{r+1}{n-r}$ (05)

We have $\frac{r+1}{n-r} = \frac{45}{120}$ *and* $\frac{r+2}{n-r-1} = \frac{120}{210}$ (05) (05)

i.e. $3n - 11r = 8$ and $4n - 11r = 18$

$\therefore n = 10$ (05)

Also $r = 2$ and note that

${}^{10}C_2 = 45$, ${}^{10}C_3 = 120$ and ${}^{10}C_4 = 210$ 25

(c) No! (05) 05

Suppose that it is possible for three consecutivecoefficients to be in Geometric progression.

i.e. there exists an integer r such that ${}^nC_r$, ${}^nC_{r+1}$ and ${}^nC_{r+2}$ are in Geometric progression. (05)

Then, we have $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{{}^nC_{r+1}}{{}^nC_{r+2}}$ (05)

i.e $\frac{r+1}{n-r} = \frac{r+2}{n-r-1}$ (05)

It follows that $n + 1 = 0$ (05)

This is a contradiction. 20

(03) (a)

Let $f(n) = 5^{n+1} - 2^{n+1} - 3^{n+1}$ for $n \in \mathbb{Z}^+$

When $n = 1$, $f(n) = f(1) = 5^2 - 2^2 - 3^2 = 25 - 13 = 12$

Clearly, $f(1)$ is divisible by 6. (05)

Thus the result is true for n = 1

Assume that the result is true fer $n = p \in \mathbb{Z}^+$

i.e. $f(p) = 5^{P+1} - 2^{P+1} - 3^{P+1} = 6m$ Where $m \in \mathbb{Z}^+$ (05)

Now, $f(p+1) = 5^{P+2} - 2^{P+2} - 3^{P+3}$

$= 5(6m + 2^{P+1} + 3^{P+1}) - 2^{P+2} - 3^{P+2}$

$= 30m + 5.2^{P+1} + 5.3^{P+1} - 2.2^{P+1} - 3.3^{P+1}$

$= 30m + 3.2^{P+1} + 2.3^{P+1}$ (05) (05)

Then $f(p+1) = 30m + 3.2^{P+1} + 2.3^{P+1}$

$= 6(5m + 2^P + 3^P)$

$= 6k$; Where $k \in \mathbb{Z}^+$ (05)

Thus f(p) is divisible by 6 and hence the result is true for n = p+1 (05)

By the principle of mathematical Induction (05)

$f(n) = 5^{n+1} - 2^{n+1} - 3^{n+1}$ indivisible by 6 $\forall n \in \mathbb{Z}^+$ 35

(b) (i) $2^n = (1+1)^n$

$= 1 + {}^nC_1 + \ldots\ldots + {}^nC_r + \ldots\ldots + {}^nC_n$

$= 1 + \sum_{r=1}^{n} {}^nC_r$ (05)

$\therefore \sum_{r=1}^{n} {}^nC_r = 2^n - 1$ (05)

Now, ${}^nC_2 = \frac{n(n+1)}{2} < \sum_{r=1}^{n} {}^nC_r$ (05)

$= 2^n - 1$

$< 2^n$ (05) 20

(ii) $U_r = \frac{2^{i-i} \cdot r}{(r+1)(r+2)}$

$= 2^{r-1}\left[\frac{2}{r+2} - \frac{1}{r+1}\right]$ (05)

$= \frac{2^r}{(r+2)} - \frac{2^{r-1}}{(r+1)}$ (05)

$= f(r) - f(r-1)$ where $f(r) = \frac{2^r}{r+2}$ (05) 15

$S_n = \sum_{r=1}^{n} U_r = \begin{pmatrix} [f(1) - f(0)] + [f(2) - f(1)] + \ldots\ldots + \\ [f(n-1) - f(n-2)] + [f(n) - f(n-1)] \end{pmatrix}$ (05)

$= f(n) - f(0)$ (05)

$= \frac{2^n}{(n+2)} - \frac{1}{2}$ (05) 15

$S_n = \frac{2^n}{n}\left(\frac{n}{n+2}\right) - \frac{1}{2}$

$\frac{2^n}{n} > \left(\frac{n-1}{2}\right)$ implies that $S_n > \frac{(n-1)n}{2(n+2)} - \frac{1}{2}$ (05)

Also, $\frac{(n-1)n}{2(n+2)} - \frac{1}{2} \to \infty$ as $n \to \infty$

Hence, $S_n \to \infty \notin \mathbb{R}$ as $n \to \infty$ (05)

$\therefore \lim_{n \to \infty} S_n$ does not exist in $\mathbb{R}$ (05) 15

(04) $z^3 - 1 = (z-1)(z^2 + z + 1)$ (05)

$z^3 - 1 = 0 \Leftrightarrow z - 1$ or $z^2 + z + 1 = 0$

$\Leftrightarrow z = 1$ or $z = \frac{-1 \pm \sqrt{3}\, i}{2}$ (05) 10

Let $w = \frac{-1 \pm \sqrt{3}i}{2}$

Then $w^2 = \frac{1 - 2\sqrt{3}i - 3}{4} = \frac{-1 \mp \sqrt{3}i}{2}$ (10)

Hence if one complex root is w, then the other is w^2 10

Also $\frac{1}{1+\omega} = \frac{1}{\left(\frac{1}{2} \pm \frac{\sqrt{3}i}{2}\right)} \times \frac{\left(\frac{1}{2} \mp \frac{\sqrt{3}i}{2}\right)}{\left(\frac{1}{2} \mp \frac{\sqrt{3}i}{2}\right)} = \left(\frac{1}{2} \mp \frac{\sqrt{3}i}{2}\right)$ (05)

By the previons result

$\frac{1}{1+w^2} = \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$ (05)

since $w^3 = 1$, $\frac{1}{1+w^3} = \frac{1}{2}$ (05)

Hence $\mathrm{Re}\left(\frac{1}{1+W^r}\right) = \frac{1}{2}$, for $r = 1, 2, 3$ 15

The point that represent the complex number $\frac{1}{1+W^r}$ for r = 1, 2, 3 in the Argand diagram lie on the straight line

$\text{Re}z = \frac{1}{2}$ (05) [05]

$(z_1 + wz_2 + w^2z_3)(z_1 + w^2z_2 + wz_3)$ (05)

$= z_1^2 + w^3z_2 + w^3z_3 + (w + w^2)z_1z_2 + (w + w^2)z_1z_3$ (05)

$+(w^2 + w^4)z_2z_3$

$= z_1^2 + z_2^2 + z_3^3 - z_1z_2 - z_2z_3 - z_3z_1$

since $w^3 = 1, w + w^2 = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2} - \frac{1}{2} \mp \frac{\sqrt{3}i}{2} = -1$

and $w^4 = w$ (10)

Hence $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$

$\Leftrightarrow (z_1 + wz_2 + w^2z_3)(z_1 + w^2z_2 + wz_3) = 0$ (05)

$\Leftrightarrow z_1 = -wz_2 - w^2z_3$ or $z_1 = -w^2z_2 - wz_3$ (05) [30]

Let $z_1 = -wz_2 - w^2z_3$

Then $z_1 - z_2 = -wz_2 - w^2z_3 - z_2$

$= -(1 + w)z_2 - w^2z_3$

$= w^2z_2 - w^2z_3$ ($\because 1 + w + w^2 = 0$)

(05) (05)

$= w^2(z_2 - z_3) \rightarrow (01)$

and $z_1 - z_2 = -wz_2 - w^2z_3 - z_3$

$= -wz_2 - (1 + w^2)z_3$

$= -wz_2 + wz_3$ $(1 + w + w^2 = 0)$

$= -w(z_2 - z_3) \rightarrow (02)$ (05)

Now since $|w| = |w^2| = 1$ (05)

We have from (01) and (02) that

$|z_1 - z_2| = |z_2 - z_3|$ *and* $|z_1 - z_3| = |z_2 - z_3|$

$\therefore |z_1 - z_2| = |z_2 - z_3| = |z_1 - z_3|$ (05)

z_1, z_2, z_3 represent vertices of an equilateral triangle.

next we let $z_1 = -w^2z_2 - wz_3$

By interchanging z_2 and z_3 in the above case, we get

$|z_1 - z_3| = |z_3 - z_2| = |z_1 - z_2|$ Again z_1, z_2, z_3 represen vertices of an equilateral triangle. (05) [30]

Aliter

Let z_1, z_2 and z_3 represent point P_1, P_2 and P_3

As before $z_1 - z_2 = -w(z_2 - z_3)$ (10)

$\therefore P_1\hat{P}_2P_3 = \text{Arg}\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \text{Arg}\ w = \frac{\pi}{3}$ (05) (05)

and $|z_1 - z_2| = |z_2 - z_3|$ (05)

$P_1P_2 = P_3P_2$

$\therefore$ $P_1P_2P_3$ is an equilateral triangle. (05) [30]

(05) (*a*)

$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ (05)

$= \lim_{\Delta x \to 0} \frac{\tan(x + \Delta x) - \tan x}{\Delta x}$ (05)

$= \lim_{\Delta x \to 0} \frac{\frac{\sin(x + \Delta x)}{\cos(x + \Delta x)} - \frac{\sin x}{\cos x}}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x)\cos x - \cos(x + \Delta x)\sin x}{\cos(x + \Delta x)\cos x}$ (05)

$= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x - x)}{\Delta x} \cdot \frac{1}{\cos(x + \Delta x)\cos x}$ (05)

$= 1 \cdot \frac{1}{\cos^2 x}$ $\left[\because \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1\right]$

$= \sec^2 x$ (05)

Hence the derivative of tan *x* with respect to *x* is $\sec^2 x$. [25]

Aliter

$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ (05)

$= \lim_{\Delta x \to 0} \frac{\tan(x + \Delta x) - \tan x}{\Delta x}$ (05)

$= \lim_{\Delta x \to 0} \frac{\tan(x + \Delta x - x)}{\Delta x}\left[1 + \tan(x + \Delta x)\tan x\right]$ (05)

$= \lim_{\Delta x \to 0} \frac{\tan \Delta x}{\Delta x}\left[1 + \tan(x + \Delta x) \cdot \tan x\right]$

$= 1 \cdot (1 + \tan^2 x)$ $\left[\because \lim_{\Delta x \to 0} \frac{\tan(\Delta x)}{\Delta x} = 1\right]$ (05)

$= \sec^2 x$ (05)

Hence the derivative of tan *x* with respect to *x* is $\sec^2 x$. [25]

$\frac{d}{dx}\tan\left(\sin^{-1}x\right) = \sec^2\left(\sin^{-1}x\right)\frac{1}{\sqrt{1-x^2}}$ (10) [10]

(b) $u = \ln(\cos x)$

Then $\frac{du}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x$ (05)

By the chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$ (05)

$$= -\tan x\frac{dy}{du}$$

$\frac{d^2y}{dx^2} = -\sec^2 x\cdot\frac{dy}{du} - \tan x\frac{d}{dx}\left(\frac{dy}{du}\right)$ (05)

$= -\sec^2 x\cdot\frac{dy}{dx}\cdot\frac{dx}{du} - \tan x\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}$ (10)

$= -\sec^2 x\cdot\frac{dy}{dx}\cdot\frac{1}{(-\tan x)} - \tan x\frac{d^2y}{du^2}(-\tan x)$ (05)

$\Rightarrow -\tan x\frac{d^2y}{dx^2} = \sec^2 x\frac{dy}{dx} + \tan^3 x\frac{d^2y}{du^2} \rightarrow (A)$

Multiplication by $\cos^3 x$ *yields*

$\sin^3 x\frac{d^2y}{dx^2} = \cos^2 x\cdot\sin x\frac{d^2y}{dx^2} - \cos x\frac{dy}{dx}$ (05) [35]

(c) $\frac{dx}{dt} = \frac{a}{2}\left(1-\frac{1}{t^2}\right)$, $\frac{dy}{dt} = a\left(1+\frac{1}{t^2}\right)$ (05)

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\left(t^2+1\right)}{\left(t^2-1\right)}$ (05)

∴ The slope of the normal L at the point with Parameter value $t_0 = \frac{-\left(t_0^2-1\right)}{2\left(t_0^2+1\right)}$ (05)

The equation of L :

$y - a\left(t_0 - \frac{1}{t_0}\right) = \frac{-\left(t_0^2-1\right)}{2\left(t_0^2+1\right)}\left\{x - \frac{a}{2}\left(t_0+\frac{1}{t_0}\right)\right\}$ (05)

$4\left(t_0^2+1\right)t_0y - 4a\left(t_0^4-1\right) = -2\left(t_0^2-1\right)t_0x + a\left(t_0^4-1\right)$

$2\left(t_0^2-1\right)t_0x + 4\left(t_0^2+1\right)t_0y = 5a\left(t_0^4-1\right)$ (05)

This goes through $(-13a, 0)$ if and only if

$-26\left(t_0^2-1\right)t_0a = 5a\left(t_0^4-1\right)$

$\left(t_0^2-1\right)\left(5t_0^2+26t_0+5\right) = 0$

$(t_0-1)(t_0+1)(5t_0+1)(t_0+5) = 0$

$t_0 = \pm1$ or $t_0 = -5$ or $t_0 = -\frac{1}{5}$ (05)

Hence 4 normals can be drawn from (-13a, 0) and the parametric value of the feet of the normals are ±1, -5 and $-\frac{1}{5}$. [30]

(06) (a)

We note that $\frac{1}{x^2-a^2} = \frac{1}{2a}\left[\frac{1}{(x-a)} - \frac{1}{(x+a)}\right]; a \neq 0$ (05)

Thus we have

$\frac{1}{\left(x^2-a^2\right)^2} = \frac{1}{4a^2}\left[\frac{1}{(x-a)^2} - \frac{2}{(x-a)(x+a)} + \frac{1}{(x+a)^2}\right]$ (05)

$= \frac{1}{4a^2}\frac{1}{(x-a)^2} + \frac{1}{4a^2}\left(\frac{1}{x+a^2}\right) - \frac{1}{2a^2}\cdot\frac{1}{2a}\left(\frac{1}{(x-a)} - \frac{1}{(x+a)}\right)$ (05)

$= \frac{1}{4a^2}\frac{1}{(x-a)^2} + \frac{1}{4a^2}\frac{1}{(x+a)^2} - \frac{1}{4a^3}\frac{1}{(x-a)} + \frac{1}{4a^3}\cdot\frac{1}{(x+a)}$ (05)

[20]

Aliter.

$\frac{1}{\left(x^2-a^2\right)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x+a)} + \frac{D}{(x+a)^2}$ (05)

$\Rightarrow 1 = A(x+a)^2(x-a) + B(x+a)^2 + C(x-a)^2(x+a) + D(x-a)^2$ (05)

$\Rightarrow 1 = (A+C)x^3 + [a(A-C) + (B+D)]x^2 + [2a(B-D) - a^2(A+C)]x + a^3(C-A) + a^2(B+D)$

$\Rightarrow$ $A + C = 0 \rightarrow (01)$

$a(A-C) + (B+D) = 0 \rightarrow (02)$

$2(B-D) = a(A+C) \rightarrow (03)$

$B + D = \frac{1}{a^2} + a(A-C) \rightarrow (04)$ (05)

$\Rightarrow B = D,\ B = aC,\ C = \frac{1}{4a^3}$ (05)

$A = -C$, Here $a \neq 0$

$\Leftrightarrow$ Hence,

$$\frac{1}{(x^2-a^2)^2} = -\frac{1}{4a^3}\cdot\frac{1}{(x-a)} + \frac{1}{4a^2}\cdot\frac{1}{(x-a)^2} + \frac{1}{4a^3}\cdot\frac{1}{(x+a)} + \frac{1}{4a^2}\cdot\frac{1}{(x+a)^2}$$ [20]

Now $\int \frac{dx}{(x^2-a^2)^2}$

$$= \left\{\begin{array}{ll} \frac{1}{4a^2}\int\frac{dx}{(x-a)^2} + \frac{1}{4a^2}\int\frac{dx}{(x+a)^2} - \frac{1}{4a^3} \\ \int\frac{dx}{(x-a)} + \frac{1}{4a^3}\int\frac{dx}{(x+a)} \end{array}\right\} : (05)$$

$$= \left(\begin{array}{ll} -\frac{1}{4a^2}\cdot\frac{1}{(x-a)} - \frac{1}{4a^2}\cdot\frac{1}{(x+a)} - \frac{1}{4a^3} \\ \ell n|x-a| + \frac{1}{4a^3}\ell n|x+a| + C \end{array}\right) (20)$$

Here $a \neq 0$ and c is an arbitrary constant.

$$= -\frac{1}{2a^2}\cdot\frac{1}{(x^2-a^2)} + \frac{1}{4a^3}\ell n\left|\frac{x+a}{x-a}\right| + C$$ [25]

(b) (i) $y = 2^x\ (> 0)$

$\Leftrightarrow \ln y = x \ln 2$ (05)

$\frac{1}{y}\frac{dy}{dx} = \ln 2$ (05)

$\Rightarrow \frac{dy}{dx} = y(\ln 2)$ (05)

$\Rightarrow \frac{d}{dx}\left(\frac{y}{\ln 2}\right) = y$

i.e. $\frac{d}{dx}\left(\frac{2^x}{\ln 2}\right) = 2^x \rightarrow (1)$ [15]

(ii) Using (1)

$\int 2^t\,dx = \frac{2^t}{\ln 2} + C$ (= *arbitrary constant*)

(05) (05) [10]

(iii) $\int_{-1}^{1} 2^{\sqrt{x+1}}\,dx = I$ (*say*)

Let $t = \sqrt{x+1}$ (05) $t = 0 \rightarrow \sqrt{2}$

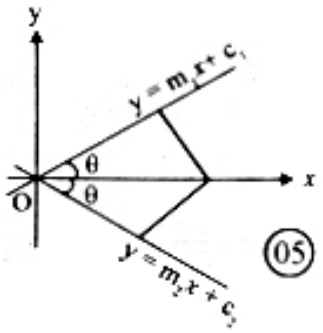
Then $dt = \frac{dx}{2t}$ (05)

$I = 2\int_0^{\sqrt{2}} t\ 2^t\ dt$ (05)

$= \int_0^{\sqrt{2}} t\ 2^t\ dt = \left[\frac{t\ .2^t}{\ln 2} - \frac{2^t}{(\ln 2)^2}\right]_0^{\sqrt{2}}$ (10)

$I = 2^{\sqrt{2}+1}\left[\frac{\sqrt{2}}{\ln 2} - \frac{1}{(\ln 2)^2}\right] + \frac{2}{(\ln 2)^2}$ (05) [30]

(07) (a)



(05)

$\frac{y - m_1x - c_1}{\sqrt{1+m_1^2}} = \pm\frac{(y - m_2x - c_2)}{\sqrt{1+m_2^2}}$ (10)

$l_1 : (a_2 - a_1)y = (a_2m_1 - a_1m_2)x + (a_2c_1 - a_1c_2)$

$l_2 : (a_2 + a_1)y = (a_2m_1 + a_1m_2)x + (a_2c_1 + a_1c_2)$

for both ℓ_1 and l_2, where $a_1 = \sqrt{1+m_1^2}$, $a_2 = \sqrt{1+m_2^2}$ [15]

Case (i) $m_1 = -m_2$

$m_1 = -m_2 \Leftrightarrow a_1 = a_2 \quad (\because m_1 \neq m_2)$

$\ell_1 : x = constants$ (05)

$\ell_2 : y = constants$

$\therefore l_1 \perp l_2$

Case (ii) $m_1 \neq -m_2$

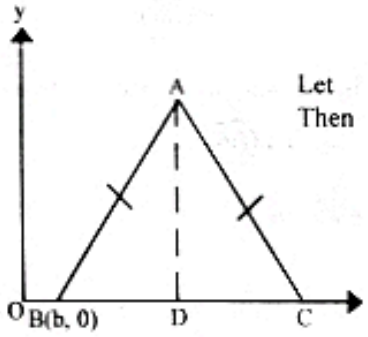
$l_1 \Rightarrow y = M_1x + k_1$, *where* $M_1 = \dfrac{a_2m_1 - a_1m_2}{a_2 - a_1}$ = *slope of* l_1 (05)

$l_2 \Rightarrow y = M_2x + k_2$, *where* $M_2 = \dfrac{a_2m_1 + a_1m_2}{a_2 + a_1}$ = *slope of* l_2 (05)

$\therefore\ M_1 \cdot M_2 = \dfrac{a_2^2m_1^2 - a_1^2m_2^2}{a_2^2 - a_1^2} = \dfrac{m_1^2 - m_2^2}{m_2^2 - m_1^2} = -1$

$\therefore\ l_1 \perp l_2$ (05) [20]

(b)



Let AD = h
Then 6h = 18

$A \equiv (a,3)$, $B \equiv (b,0)$ *and* $C = (b+6,0)$;
BD = DC = 3 = a - b (05) [05]

Aliter

$AB^2 = (a - b)^2 + 9 = (a - b - 6)^2 + 9 = AC^2$ (05)

$\Rightarrow$ $a - b = 3$ [05]

(i) Equations of AB : $y = x - b$ (05)
and AC : $y = -x + b + 6$ (05) [10]

(ii) Bisectors of the angle B (=B)

$l_1 : y\left(1 - \sqrt{2}\right) = x - b$ (05)

$l_2 : y\left(1 + \sqrt{2}\right) = x - b$ (05)

$\left[y\left(1+\sqrt{2}\right) - x + b\right]_{(a,3)} = 3\sqrt{2} > 0$

$\left[y\left(1+\sqrt{2}\right) - x + b\right]_{(b+6,0)} = -6 < 0$

$\therefore$ Internal bisector of B is l_2

Bisectors of the angle C (=C)

$l_3 : y = -\left(\sqrt{2} - 1\right)x + \left(\sqrt{2} - 1\right)(b + 6)$

$l_4 : y = \left(\sqrt{2} + 1\right)x - \left(\sqrt{2} + 1\right)(b + 6)$

$\left[y + \left(\sqrt{2} - 1\right)x - \left(\sqrt{2} - 1\right)(b + 6)\right]_{A\,\equiv(a,3)} = 6 - 3\sqrt{2} > 0$ (05)

$\left[y + \left(\sqrt{2} - 1\right)x - \left(\sqrt{2} - 1\right)(b + 6)\right]_{B\,\equiv(b,0)} = -6\left(\sqrt{2} - 1\right) < 0$

$\therefore$ Internal bisector of $\hat{C}$ is l_3 (05) [20]

Note that AB $\perp$ AC ; AB = AC

$\therefore A\hat{B}C = \dfrac{\pi}{4}$

$\Rightarrow (l_2\ \underline{OX}) = \dfrac{\pi}{8}$ (05)

$\Rightarrow$ *slope of* $l_2 = \tan\left(\dfrac{\pi}{8}\right)$

$\Rightarrow \tan\left(\dfrac{\pi}{8}\right) = \dfrac{1}{1+\sqrt{2}}$ (05) [10]

Internal bisector of the angle A is AD and its equation is given by $x = b + 3$ (05)

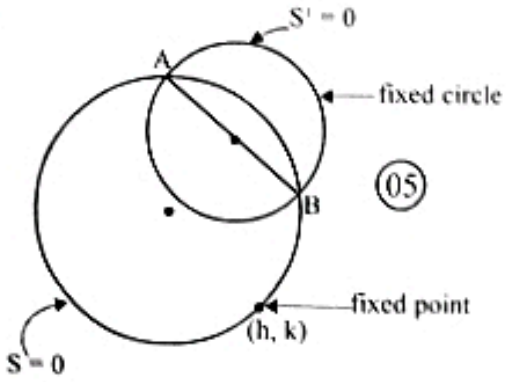
$l_2 : y = \left(\sqrt{2} - 1\right)(x - b)$ $\rightarrow$ (01)

$l_3 : y = \left(\sqrt{2} - 1\right)(-x + b + 6)$ $\rightarrow$ (02)

Solving (01) and (02), we get the coordinates of the point of Intersection of l_2 and l_3 as $x = b + 3$, $y = 3\left(\sqrt{2} - 1\right)$; where b is a parameter (05) (05)

Thus l_2, l_3 and AD are concurrent lines and the locus of the point of concurrency is given by $y = 3\left(\sqrt{2} - 1\right)$ = Constant (05) [20]

(08)



05

Then equation of AB is $S^1 - S = 0$

i.e. $2(g^1 - g)x + 2(f^1 - f)y + c^1 - c = 0$ (05)

Since this goes ghrough $(-g^1, -f^1)$, we have. (05)

$2(g^1 - g)g^1 + 2(f^1 - f)f^1 + c^1 - c = 0 \rightarrow (01)$ (05)

Since S = 0 goes through (h, k)

$h^2 + k^2 + 2gh + 2fk + c = 0 \quad \rightarrow (02)$ (05)

(01) and (02) ⇨

$-2g^{12} + 2gg^1 - 2f^{12} + 2ff^1 + c^1 + h^2 + k^2 + 2gh + 2fk = 0$ (05)

$(2g^1 + 2h)g + (2f^1 + 2k)f - 2g^{12} - 2f^{12} + h^2 + k^2 + c^1 = 0$

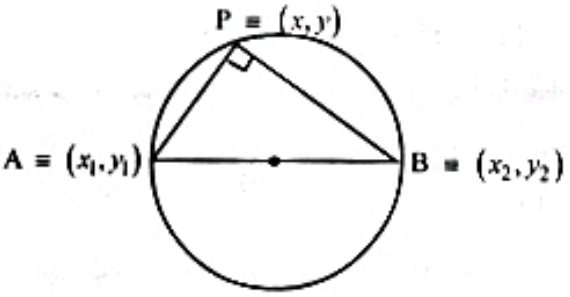
$2(g^1 + h)(-g) + 2(f^1 + k)(-f) + 2g^{12} + 2f^{12} - h^2 - k^2 - c^1 = 0$ (05)

∴ Then the centre of S = 0 lies on the straight line.

⇨ $2(g^1 + h)x + 2(f^1 + k)y + 2g^{12} + 2f^{12} - h^2 - k^2 - c^1 = 0$ (05)

Since f^1, g^1, h, k and c^1 are constants, this is a fixed straight line. (05)

45



Let $P \equiv (x, y)$ be any point on the circle.

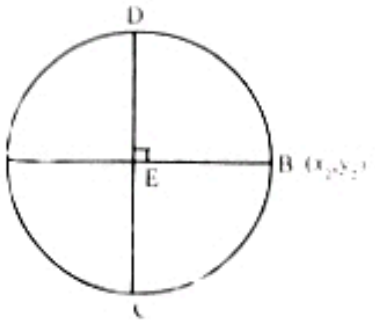
Then $AP^2 + PB^2 = AB^2$ (05)

$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ (10)

$x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2 = 0$ (05)

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

20



Let $E \equiv (x_0, y_0)$ be the centre.

Since E is the mid point of AB, we have

$x_0 = \frac{x_1 + x_2}{2}$ and $y_0 = \frac{y_1 + y_2}{2}$ (05)

Let $C \equiv (x_0 + \lambda, y_0 + \mu)$ (05)

Then, Since E is the mid-point of CD, we have

$D \equiv (x_0 - \lambda, y_0 - \mu)$ (05)

Note that $CE^2 = \lambda^2 + \mu^2$ and $CE = \frac{1}{2}AB$

$\Rightarrow \lambda^2 + \mu^2 = \frac{1}{4}\{(x_2 - x_1)^2 + (y_2 - y_1)^2\} \rightarrow (01)$ (05)

Also since $CE \perp AB$,

$\left(\frac{\mu}{\lambda}\right)\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = -1$ (05)

$\mu(y_2 - y_1) = -\lambda(x_2 - x_1) \rightarrow (02)$

$(01) \Rightarrow \lambda^2 + \lambda^2\frac{(x_2 - x_1)^2}{(y_2 - y_1)^2} = \frac{1}{4}\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]$

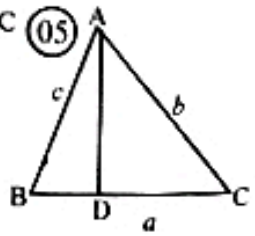
$\Rightarrow \lambda^2 = \frac{1}{4}(y_2 - y_1)^2$

$\Rightarrow \lambda = \pm\frac{1}{2}(y_2 - y_1)$ (05)

now $(02) \Rightarrow \mu = \mp\frac{1}{2}(x_2 - x_1)$ (05)

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(09) $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, Where AB = C, bc = a and AC = b in the triangle ABC (05)



Situation I

ABC is an acute-angled triangle

$AD = AB \sin B = AC \sin C$

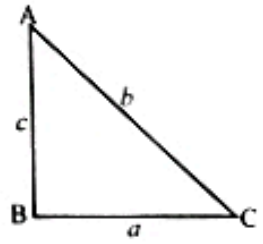
$\therefore \dfrac{\sin B}{AC} = \dfrac{\sin C}{AB}$

similarly, $\dfrac{\sin A}{AC} = \dfrac{\sin C}{AB}$ (05)

$\therefore \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Situation II

ABC is a righ- angled triangle

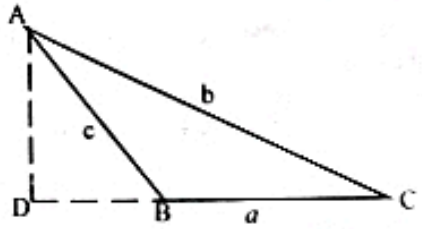


$AB = AC \sin C = AB \sin B$

$\therefore \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ (05)

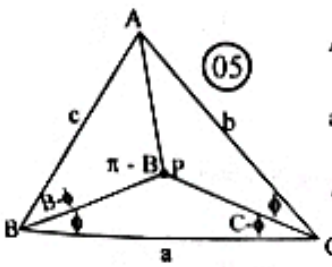
Situation III

ABC is an obtuse angled triangle



$AD = AB \sin(\pi - B) = AC \sin C$

$\therefore \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ (05) 20



(05)

$A\hat{P}B = \pi - B$, $B\hat{P}C = \pi - C$

and $C\hat{P}A = \pi - A$

$ABP\ \Delta$, $\dfrac{BP}{\sin\phi} = \dfrac{C}{\sin(\pi - B)}$

$\Rightarrow BP = C\left(\dfrac{\sin\phi}{\sin B}\right) \rightarrow (01)$ (05)

similarly, we get

$CP = a\left(\dfrac{\sin\phi}{\sin C}\right) \rightarrow (02)$

$BP = b\left(\dfrac{\sin\phi}{\sin A}\right) \rightarrow (03)$ (05)

$BCP\ \Delta$: $\dfrac{\sin(c-\phi)}{\sin\phi} = \dfrac{BP}{CP} = \dfrac{c \sin C}{a \sin B}$ (05)

$\Rightarrow \dfrac{\sin(c-\phi)}{\sin C \sin\phi} = \dfrac{C}{a \sin B}$

$\Rightarrow \dfrac{ab}{c}(\cot\phi - \cot C) = \dfrac{b}{\sin B} \rightarrow (04)$ (05)

similarly, we get

$\dfrac{ac}{b}(\cot\phi - \cot B) = \dfrac{a}{\sin A} \rightarrow (05)$

$\dfrac{bc}{a}(\cot\phi - \cot A) = \dfrac{c}{\sin C} \rightarrow (06)$ (05)

Using the sine rule, We have the required result. (05) 35

(b) $\cos x + \cos y = t$

$\sin x + \sin y = t$

$x + y + z = \pi;\ x, y, z \geq 0$

$\therefore t \geq 0$

(i) $t = 2\sin\dfrac{(x+y)}{2}\cos\dfrac{(x-y)}{2}$ (05)

$1 = 2\cos\dfrac{(x+y)}{2}\cos\dfrac{(x-y)}{2}$ (05)

$\Rightarrow t = \tan\left(\dfrac{x+y}{2}\right) \rightarrow (01)$ (05)

$\tan^{-1}(t) = \dfrac{x+y}{2}$ 15

(ii) $1 + t^2 = 2 + 2\cos(x-y) \rightarrow (02)$ (05)

$-1 \leq \cos(x-y) \leq 1$ (05)

$\therefore 0 \leq t^2 \leq 3$

$\Rightarrow 0 \leq t \leq \sqrt{3}$ (05)

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