Combined Mathematics I – 2007 Advanced Level Examination

❖ Answer six questions only.

01. (a) α and β are the roots of the equation $x^2 + bx + c = 0$. Find the quadratic equation, in terms of b and c, whose roots are $\alpha^3$and β^3. Hence, find the quadratic equation, in terms of b and c, whose roots are $\alpha^3 + \frac{1}{\beta^3}$ and $\beta^3 + \frac{1}{\alpha^3}$.

(b) $f(x)$ is a polynomial in x of degree greater than 3. When $f(x)$ is divided by $(x-1)(x-2)$ and $(x-3)$, the remainders are a, b and c respectively. By repeated application of the Remainder Theorem, show that when $f(x)$ is divided by $(x-1)(x-2)(x-3)$, the remainder can be expressed as $\lambda(x-1)(x-2) + \mu(x-1) + v$ where $\lambda,\ \mu$ and v are constants.
Find $\lambda,\ \mu$ and v in terms of a, b and c.

02. (a) A candidate sitting an examination is required to answer six questions out of twelve given under three parts A, B and C, with each part containing four questions. Find the number of different ways the candidate can select six questions if,
(i) the first question in each part is compulsory.
(ii) he cannot answer more than three questions from any part,
(iii)it is compulsory to answer at least one question from each part.

(b) State the binomial theorem for a positive integral index.
Let a, b and d be integers such that $a = b + d$. Show that $a^n - b^n(b + nd)$ is divisible by d^2 for positive integral n.
If U is the n^{th} term of an arithmetic progression whose first term is a and the common difference is d, prove that $a^n - (a-d)^{n-1}$, U is divisible by d^2.
Deduce that $7^{60} - 3^{64}$ is divisible by 16.

03. (a) By using the principle of Mathematical Induction, prove that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{34n}{105}$ is an integer, for positive integral n.

(b) Write down u_r, the r^{th} term of the series $\frac{3}{1.2}\left(\frac{1}{2}\right) + \frac{4}{2.3}\left(\frac{1}{2}\right)^2 + \frac{5}{3.4}\left(\frac{1}{2}\right)^3 + \ldots\ldots\ldots\ldots$
Find $f(r)$ such that $u_r = f(r-1) - f(r)$.
Hence, find $S_n = \sum_{r=1}^{n} u_r$
Evaluate $\lim_{n\to\infty} S_n$.

04. (a) The complex number $z_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ and $z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ are represented on an Argand diagram by the points A and B respectively. Find Arg z_1 and Arg z_2.

Given that $OACB$ is a square in the Argand diagram, where O is the origin, find the modulus and the argument of the complex number represented by C.

(b) (i) Find the least and the greatest values of $|z-3|$, subject to the condition

$\left|z - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right| \leq 2$.

(ii) Find the least value of $|z|$, subject to the condition $\arg(z-1) = \frac{\pi}{6}$.

05. (a) (i) Show that $\frac{d^r}{dx^r}(xe^x) = (x+r)e^x$ for any positive integer r.

(ii) If $y = x^2e^x$, prove that $\frac{dy}{dx} = 2xe^x + y$.

Deduce that $\frac{d^r y}{dx^r} - \frac{d^{r-1}y}{dx^{r-1}} = 2(x+r-1)e^x$.

Hence, show that $\frac{d^n y}{dx^n} = n(2x+n-1)e^x + y$, for any positive interger n.

(b) The tangent at the point $P(at^2, at^3)$, to the curve $ay^2 = x^3$, where a is a constant, meets the curve again at Q. Find the coordinates of Q in terms of t.

06. (a) Using partial fractions, find $\int \frac{x^3+1}{x(x-1)^3}dx$.

(b) Find A, B and C such that

$25\cos x + 15 \equiv A(3\cos x + 4\sin x + 5) + B(-3\sin x + 4\cos x) + C$.

Hence, find $\int \frac{25\cos x + 15}{3\cos x + 4\sin x + 5}dx$

(c) Using the method of integration by parts, show that

$$\int_0^{\frac{\pi}{2}} sin^6x\,dx = \frac{5}{6}\int_0^{\frac{\pi}{2}} sin^4x\,dx = \frac{5.3}{6.4}\int_0^{\frac{\pi}{2}} sin^2x\,dx = \frac{5\pi}{32}$$

Hence, evaluate $\int_0^{\frac{\pi}{6}} sin^6 3x\,dx$

07. Let ABC be a triangle such that $A \equiv (2,4)$ and B and C lie $y = x+1$. A line l, drawn parallel to BC cuts AM and AC at D and E respectively such that the areas of the triangels ABC and ADE are in the

ratio $9:4$. Let G be the foot of the perpendicular from A to l and M be the mirror image of G in the line AB.

(i) Find the coordinates of G and the equation of l.

(ii) Show that $AM = AG$.

Hence or otherwise, prove that, as the point B moves along the line $y = x + 1$, the point M moves on a circle which has the centre at A and the radius $\frac{\sqrt{2}}{3}$.

08. Two circles are said to intersect orthogonally when the two tangents at each point of intersection are at right angles. Find the condition for the two circles to $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ to intersect orthogonally.

Prove that the equation $x^2 + y^2 + 4x + 2\lambda y - 6 = 0$……(*)

Where λ is a parameter, represents a system of circle passing through the points $(-2 + \sqrt{10}, 0)$ and $\left(-2 - \sqrt{10}, 0\right)$.

$S = 0$ is a circle belonging to the system represented by (*). Show that there exists a unique circle $S' = 0$ belonging to the same system which is orthogonal to $S = 0$.

Find $S' = 0$ when $x^2 + y^2 + 4x + 4y - 6 = 0$.

Find also the general equation of the circle orthogonal to both $S = 0$ and $S' = 0$.

09. (a) State the sine rule, in the usual notation. P is a point inside a triangle ABC such that $< PAB =< PBC =< PCA = \varphi$.

Prove that the area of the triangle ABC is $\frac{abc}{2}\left(\frac{BP}{bc} + \frac{CP}{ac} + \frac{AP}{ab}\right)\sin\varphi$, in the usual notation.

Deduce that $\frac{1}{\sin^2\varphi} = \frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C}$

(b) Show that

(i) $2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{5}{12}\right)$

(ii) $2\tan^{-1}\left(\frac{5}{12}\right) = \tan^{-1}\left(\frac{120}{119}\right)$

(iii) $\tan^{-1}\left(\frac{120}{119}\right) - \frac{\pi}{4} = \tan^{-1}\left(\frac{1}{239}\right)$

Deduce that $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \frac{\pi}{4}$.