G.C.E. (Advanced Level) Examination - August 2007
10 - Combined Mathematics - I
Marking Scheme

(01) (a) α, and β are the roots of the equation $x^2 + bx + c = 0$ and thus, We have

$\alpha + \beta = -b$ and $\alpha\beta = C$ (05)

$\alpha^2 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$, $\alpha^3\beta^3 = c^3$ (05)

$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$ (05)

$= (-b)[b^2 - 3c]$

$= -b^3 + 3bc$ (05)

$\therefore$ The quadratic equation whose roots are α^3 and β^3 is

$x^2 + b(b^2 - 3c)x + c^3 = 0$ (05) [25]

$\alpha^3 + \frac{1}{\beta^3} = \frac{\alpha^3\beta^3 + 1}{\beta^3} = \frac{c^3 + 1}{\beta^3}$ and (05)

$\beta^3 + \frac{1}{\alpha^3} = \frac{\alpha^3\beta^3 + 1}{\alpha^3} = \frac{c^3 + 1}{\alpha^3}$ (05)

Let $y = \frac{c^3 + 1}{x}$ (05)

When $x = \alpha^3$, $y = \frac{c^3 + 1}{\alpha^3} = \beta^3 + \frac{1}{\alpha^3}$ and when $x = \beta^3$,

$y = \frac{c^3 + 1}{\beta^3} = \alpha^3 + \frac{1}{\beta^3}$

Putting $x = \frac{c^3 + 1}{y}$ in $x^2 + b(b^2 - 3c)x + c^3 = 0$ we have

$\left(\frac{c^3 + 1}{y}\right)^2 + b(b^2 - 3c)\left(\frac{c^3 + 1}{y}\right) + c^3 = 0$ (05)

i.e. $c^3y^2 + b(b^2 - 3c)(c^3 + 1)y + (c^3 + 1)^2 = 0$

Replacing y by x we have

$c^3x^2 + b(b^2 - 3c)(c^3 + 1)x + (c^3 + 1)^2 = 0$

$\therefore$ The quadratic equation, whose roots are

$\alpha^3 + \frac{1}{\beta^3}$ and $\beta^3 + \frac{1}{\alpha^3}$ is

$c^3x^2 + (b^3 - 3bc)(c^3 + 1)x + (c^3 + 1)^2 = 0$ (05) [25]

(b) Let $Q(x)$ and υ be the quotient and the remainder when

$f(x)$ is divided by $(x - 1)$

Then $f(x) \equiv Q(x)(x - 1) + \upsilon$ (05)

Let $R(x)$ and μ be the quotient and the remainder

when $Q(x)$ is divided by $(x - 2)$

Then $Q(x) \equiv R(x)(x - 2) + \mu$ (05)

$f(x) = \{R(x)(x - 2) + \mu\}(x - 1) + \upsilon$

$= R(x)(x - 2)(x - 1) + \mu(x - 1) + \upsilon$ (05)

Let $S(x)$ and λ be the quotient and remainder, When

$R(x)$ is divided by $(x - 3)$

Then $R(x) \equiv S(x)(x - 3) + \lambda$ (05)

$f(x) = \{S(x)(x - 3) + \lambda\}(x - 2)(x - 1) + \mu(x - 1) + \upsilon$

(05)

$= S(x)(x - 3)(x - 2)(x - 1) + \lambda(x - 2)(x - 1)$

$+ \mu(x - 1) + \upsilon$

$\lambda(x - 1)(x - 2) + \mu(x - 1) + \upsilon$ is the remainder

When $f(x)$ is divided by $(x - 3)(x - 2)(x - 1)$ (05) [30]

$f(1) = a = \upsilon$, (05) $f(2) = b = \mu + \upsilon$,

$f(3) = c = 2\lambda + 2\mu + \upsilon$ (05)

$\mu = b - a$, (05) , $\lambda = \frac{1}{2}(c - 2b + a)$ (05) [20]

(02) (i)

Possible combination	Number of ways, that parts can be selected	Number of ways that the questions can be selected from parts	Total
1. Three questions from any part	3	1	3
2. Two questions from one part and one question from any of the other two parts	${}^3C_1 \times {}^2C_1$ $= 6$	${}^3C_2 \times {}^3C_1$ $= 9$	54
3. One question from each part	1 (05)	${}^3C_1 \times {}^3C_1 \times {}^3C_1$ $= 27$ (05)	27 (05)

∴ If the first question in each part is compulsory, the number of different ways the candidate can select six questions = 3 + 54 + 27 = 84 (05) [20]

Aliter :

If the first question in each part is compulsory, then the other 3 questions have to be chosen from among the other 9 questions. (10)

The number of ways these 3 questions can be chosen from 9 questions $= {}^9C_3 = \frac{9 \cdot 8 \cdot 7}{3!} = 84$ (05) (05) [20]

(ii)

Possible combination	Number of ways, that parts can be selected	Number of ways that the questions can be selected from parts	Total
1. Three questions from any two parts	${}^3C_2 = 3$	${}^4C_3 \times {}^4C_3$ $= 16$	48
2. Three questions from one part, two questions from one part and one question from the other.	$3! = 6$	${}^4C_3 \times {}^4C_2 \times {}^4C_1$ $= 96$	576
3. Two question from each part	1 (05)	${}^4C_2 \times {}^4C_2 \times {}^4C_2$ $= 216$ (05)	216 (05)

∴ If it is compulsory to answer at least one questions from each part the number of different ways the candidate can select six questions = 48 + 576 + 216 = 840 (05) [20]

(b) Binomial theorem

$(x + y)^n = x^n + {}^nC_1 x^{n-1} y + \ldots\ldots + {}^nC_r x^{n-r} y^r + \ldots\ldots + y^n$ (05)

Where

${}^nC_r = \frac{n!}{r!(n-r)!}$ (05) [10]

$a^n = (b + d)^n = b^n + {}^nC_1 b^{n-1} d + {}^nC_2 b^{n-2} d^2 + \ldots\ldots + {}^nC_r b^{n-r} d^r + \ldots\ldots\ldots {}^nC_n \cdot d^n$ (05)

$a^n - b^{n-1}(b + nd) = d^2 \{{}^nC_2 b^{n-2} + \ldots\ldots\ldots {}^nC_r b^{n-r} d^{r-2} + \ldots\ldots + d^{n-2}\}$

Thus $a^n - b^{n-1}(b + nd)$ is divisible by d^2 for positive intergral n (05) [10]

$u = a + (n - 1)d = (a - d) + nd$

$a^n - (a - d)^{n-1} u = a^n - (a-d)^{n-1} \{(a - d) + nd\}$ (05)

$= a^n - b^{n-1}(b + nd)$ Where $a = b + d$

By first part $a^n - (a - d)^{n-1}$, u is divisible by d^2 (05) [10]

$7^{60} - 3^{64} = 7^{60} - (7-4)^{60-1}$ (05) $\{(7-4) + 60 \times 4\}$

By above part this is divisible by $4^2 = 16$ (05) [10]

Aliter :

Let $\alpha = 7$, $n = 60$ and $d = 4$ (05)

Then $u = 7 + 59 \times 4 = 243 = 3^5$

$a^n - (a-d)^{n-1}\, u = 7^{60} - 3^{64}$ is divisible by $4^2 = 16$ (05)

[10]

(03) (a)

Let $f(n) = \frac{n^7}{7} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{34n}{105}$

When $n = 1$, $f(1) = \frac{1}{7} + \frac{1}{5} + \frac{1}{3} + \frac{34}{105}$

$= \frac{15 + 21 + 35 + 34}{105} = \frac{105}{105} = 1$ (05)

Thus the result is true for $n = 1$

suppose that the result is true for $n = p$

i.e. $f(P) = \frac{P^7}{7} + \frac{P^5}{5} + \frac{P^3}{3} + \frac{34P}{105}$ is an integer (05)

When $n = P + 1$,

$f(P+1) = \frac{(P+1)^7}{7} + \frac{(P+1)^5}{5} + \frac{(P+1)^3}{3} + \frac{34(P+1)}{105}$ (05)

$= f(p) + f(1) + \frac{7P^6 + 21P^5 + 35P^4 + 35P^3 + 21P^2 + 7P}{7}$ (05)

(05) $\quad \frac{+5P^4 + 10P^3 + 10P^2 + 5P}{5}$ (05) $+ \frac{3P^2 + 3P}{3}$ (05)

Thus f(P + 1) is an integer and hence the result is true fer $n = P + 1$

By the principle of Mathematical Induction,

$\frac{n^7}{7} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{34n}{105}$ is an integer, for positive integral n. (05)

[40]

(b) $U_r = \frac{(r+2)}{r(r+1)} \left(\frac{1}{2}\right)^r$ (10) [10]

$U_r = \frac{r+2}{r(r+1)} \left(\frac{1}{2}\right)^r = \frac{1}{r} \left(\frac{1}{2}\right)^{r-1}$ (05) $- \frac{1}{(r+1)} \left(\frac{1}{2}\right)^r$ (05)

$= f(r-1) - f(r)$. Where $f(r) = \frac{1}{(r+1)} \left(\frac{1}{2}\right)^r$ (05) [15]

When $r = 1$, $U_1 = f(0) - f(1)$

$r = 2$, $U_2 = f(1) - f(2)$ (05)

.....

.....

.....

$r = n - 1$, $U_{n-1} = f(n-2) - f(n-1)$

$r = n$, $U_n = f(n-1) - f(n)$ (05)

$\sum_{r=1}^{n} U_r = f(0) - f(n)$ (05)

$= 1 - \frac{1}{(n+1)} \left(\frac{1}{2}\right)^n$ (05)

i.e. $S_n = \sum_{r=1}^{n} 1 = 1 - \frac{1}{(n+1)} \left(\frac{1}{2}\right)^n$ [20]

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{r=1}^{n} U_r = \lim_{n\to\infty} \left\{1 - \frac{1}{(n+1)} \left(\frac{1}{2}\right)^n\right\} = 1$

(05) (05) (05)

[15]

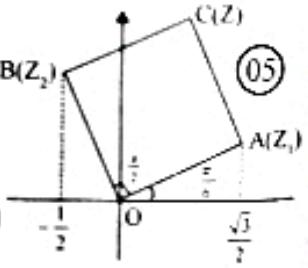
(04) (a) $z_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ (05)

$\Rightarrow Arg(z_1) = \frac{\pi}{6}$ (05) [10]

$z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right)$ (05)

$\Rightarrow \text{Arg}(z_2) = \frac{2\pi}{3}$ (05) [10]

Let the complex number represented by the point C be, (05)



(05)

$OC^2 = OA^2 + AC^2 = OA^2 + OB^2$

i.e. $|z|^2 = |Z_1|^2 + |Z_2|^2 = 2$ (05)

$\because |z_1| = |z_2| = 1$

$\therefore |z| = \sqrt{2}$ (05)

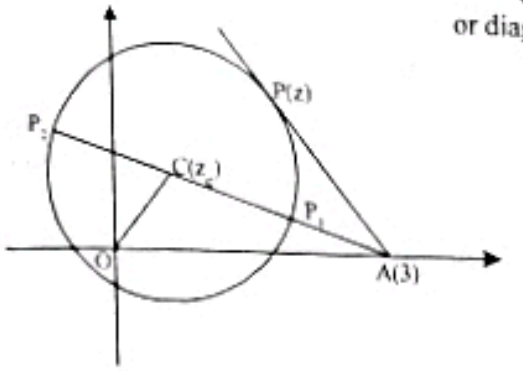
$\hat{AOC} = \frac{\pi}{4}$ (05)

$\therefore \text{Arg}(z) = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$ (05) [30]

(b) (i) Let $z_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and P be any point in the Argand diagram which represents the complex number z.

If $|z - z_0| \le 2 \Rightarrow CP \le 2$ then the point should lie in or on the circle with radius 2 and centre at C.

$|z - 3| = AP$ (10)



or diagram.

The least and the greatest values of $|z - 3|$, subject to the condition

$\left|z - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right| \le 2$ Occur at the point P_1 and P_2 (05) (05)

respectively.

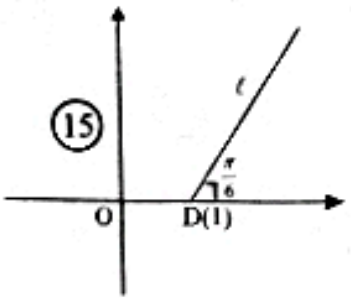
$AC = \sqrt{\left(\frac{1}{2} - 3\right)^2 + \frac{3}{4}} = \sqrt{7}$ (05)

$\therefore AP_1 = \sqrt{7} - 2$ *and* $AP_2 = \sqrt{7} + 2$

The least value $\sqrt{7} - 2$ and the greatest value $\sqrt{7} + 2$ (05) (05)

35

(ii) ℓ represents the line $\text{Arg}(z - 1) = \frac{\pi}{6}$



15

(05) **(a)** (i) $\frac{d}{dx}\left(xe^x\right) = xe^x + e^x = (x+1)e^x$ (05)

Thus the result is true for r = 1

suppose that the result is true for r = p

i.e. $\frac{d^p}{dx^p}\left(xe^x\right) = (x+p)e^x$ (05)

When r = P + 1, $\frac{d^{p+1}\left(xe^x\right)}{dx^{p+1}} = \frac{d}{dx}(x+p)e^x$ (05)

$= (x+P)e^x + e^x$

$= \{x + (p+1)\}e^x$ (05)

Thus the result is true for r = P + 1

By the principle of Mathematical Induction.

$\frac{d^r\left(xe^x\right)}{dx^r} = (x+r)e^x$ for any positive integer r. (05)

25

(ii) $\frac{dy}{dx} = \frac{d\left(x^2e^x\right)}{dx} = 2xe^x + x^2e^x = 2xe^x + y$ (05) (05)

10

$\frac{d^r y}{dx^r} = \frac{d^{r-1}}{dx^{r-1}}\left(2xe^x + y\right) = \frac{2d^{r-1}\cdot xe^x}{dx^{r-1}} + \frac{d^{r-1}y}{dx^{r-1}}$ (05)

$\frac{d^r y}{dx^r} - \frac{d^{r-1}y}{dx^{r-1}} = 2(x+r-1)e^x$ by first part (05)

10

When r = 1, $\frac{dy}{dx} - y = 2xe^x$

r = 2, $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2(x+1)e^x$ (05)

.....

.....

.....

r = n - 1, $\frac{d^{n-1}y}{dx^{n-1}} - \frac{d^{n-2}y}{dx^{n-2}} = 2(x+n-2)e^x$

r = n, $\frac{d^n y}{dx^n} - \frac{d^{n-1}y}{dx^{n-1}} = 2(x+n-1)e^x$ (05)

$\frac{d^n y}{dx^n} - y = 2nxe^x + 2e^x \sum_{r=1}^{n-1} r$ (05)

$= \{2nx + n(n-1)\}e^x = n(2x+n-1)e^x$ (05)

i.e. $\frac{d^n y}{dx^n} = n(2x+n-1)e^x + y$ for any positive integer n. (05)

25

(b) Parametric coordinates of (x, y) and (at^2, at^3)

$\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 3at^2$ so $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t}{2}$ (05)

(05)

Let $Q = \left(aT^2, aT^3\right)$

The gradient of $PQ = \frac{a\left(T^3 - t^3\right)}{a\left(T^2 - t^2\right)} = \frac{T^2 + Tt + t^2}{T + t} = \frac{3}{2}t$ (05)

$2T^2 - tT - t^2 = 0$

$(2T + t)(T - t) = 0 \Rightarrow T = -\frac{t}{2} \quad \because T \neq t$ (05)

$Q = \left(\frac{1}{4}at^2, -\frac{1}{8}at^3\right)$ (05)

(05)

30

(06) (a) $\frac{x^3 + 1}{x(x-1)^3} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$ (05)

$x^3 + 1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$

When $x = 0$ We have $A = -1$

$x = 1$ We have $D = 2$

$x = 2$ We have $9 = A + 2B + 2C + 2D$

$B + C = 3$

$x = -1$ We have $0 = -8A - 4B + 2C - D$

$2B - C = 3$

$B + C = 3$ and $2B - C = 3 \Rightarrow B = 2$ and $C = 1$

(10)

$\frac{x^3 + 1}{x(x-1)^3} = -\frac{1}{x} + \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3}$

$\int \frac{x^3 + 1}{x(x-1)^3}\,dx = -\int \frac{dx}{x} + 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} + 2\int \frac{dx}{(x-1)^3}$ (05)

$= -\ln|x| + 2\ln|x-1| - \frac{1}{(x-1)} - \frac{1}{(x-1)^2} + C$ (05)

(05)

30

(b) $25\cos x + 15 = A(3\cos x + 4\sin x + 5) + B(-3\sin x + 4\cos x) + C$,

comparing coefficients of cos terms we have

$25 = 3A + 4B$

comparing coefficients of sin terms we have

$0 = 4A - 3B$

comparing constant terms we have $15 = 5A + C$

$A = 3$, $B = 4$ and $C = 0$ (15)

$\int \frac{25\cos x + 15}{3\cos x + 4\sin x + 5}\,dx = 3\int dx + 4\int \frac{-3\sin x + 4\cos x}{3\cos x + 4\sin x + 5}\,dx$ (10)

$= 3x + 4\ln\left|3\cos x + 4\sin x + 5\right| + C$

25

(c) $\int_0^{\frac{\pi}{2}} \sin^6 x\,dx = \int_0^{\frac{\pi}{2}} \sin^5 x\,d(-\cos x)$

$= \left[-\cos x \cdot \sin^5 x\right]_0^{\frac{\pi}{2}} + 5\int_0^{\frac{\pi}{2}} \sin^4 x\cos^2 x\,dx$ (05)

$= 5\int_0^{\frac{\pi}{2}} \sin^4 x\,dx - 5\int_0^{\frac{\pi}{2}} \sin^6 x\,dx$ (05)

$\int_0^{\frac{\pi}{2}} \sin^6 x\,dx = \frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4 x\,dx$

$\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \int_0^{\frac{\pi}{2}} \sin^3 x\,d(-\cos x)$

$= \left[-\cos x \sin^3 x\right]_0^{\frac{\pi}{2}} + 3\int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\,dx$ (05)

$= 3\int_0^{\frac{\pi}{2}} \sin^2 x\,dx - 3\int_0^{\frac{\pi}{2}} \sin^4 x\,dx$ (05)

$\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

$\because \int_0^{\frac{\pi}{2}} \sin^6 x\,dx = \frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{5.3}{6.4}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ (05)

$\therefore \int_0^{\frac{\pi}{2}} \sin^6 x\,dx = \frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{5.3}{6.4} \times \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{5\pi}{32}$ (05)

30

Let $y = 3x$, Then $dy = 3dx$, $y = 0$ as $x = 0$ and $y = \pi/2$ as $x = \pi/6$ (05)

$\therefore \int_0^{\frac{\pi}{6}} \sin^6 3x\, dx = \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^6 y\, dy = \frac{1}{3} \times \frac{5\pi}{32} = \frac{5\pi}{96}$ (05) (05)

15

(07) (i)

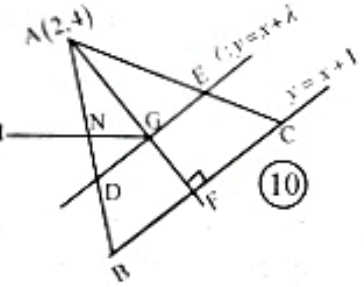
Δ ABC and Δ ADE are

similar $\Rightarrow \frac{AB}{AD} = \frac{BC}{DE}$ (05)

Δ ABF and Δ ADG are

similar $\Rightarrow \frac{AB}{AD} = \frac{AF}{AG}$ (05)

Hence $\frac{BC}{DE} = \frac{AF}{AG}$ (05)



10

Now $\frac{\Delta ABC}{\Delta ADE} = \frac{\frac{1}{2}BC \cdot AF}{\frac{1}{2}DE \cdot AG} = \frac{AF^2}{AG^2} = \frac{9}{4}$

$\Rightarrow \frac{AF}{AG} = \frac{3}{2}$ (05)

Let P(x, y) be any point on AF

Then $\frac{y-4}{x-2} = -1$ (05)

Writing $\frac{y-4}{-1} = \frac{x-2}{1} = t\,(say)$

$\Rightarrow$ $x = 2 + t$ and $y = 4 - t$

on AF is on the form $(2 + t)$ $(4 - t)$

$\therefore$ any point. If P is the Point F then

$2 + t - 4 + t + 1 = 0 \Rightarrow t = \frac{1}{2}$ (10)

Hence $F = \left(\frac{5}{2}, \frac{7}{2}\right)$

Let $G = (x_0\ y_0)$ (05)

AG : GF = 2 : 1 we have (05)

$x_0 = \frac{2 \times \frac{5}{2} + 2}{3}$ and $y_0 = \frac{2 \times \frac{7}{2} + 4}{3}$ (05)

$\therefore G = \left(\frac{7}{3}, \frac{11}{3}\right)$

60

The equation of ℓ is $y = x + \lambda$ and passes through G So $\lambda = \frac{4}{3}$ (05)

$\therefore$ the equation of ℓ is $y = x + \frac{4}{3}$ (05) 10

(ii) $AG^2 = AN^2 + NG^2 = AN^2 + NM^2 = AM^2$ (05) (05)

$\therefore$ AG = AM (05) 15

$AG = \sqrt{\left(2 - \frac{7}{3}\right)^2 + \left(4 - \frac{11}{3}\right)^2} = \frac{\sqrt{2}}{3}$ (05)

i.e. AM is of constant lengh and hence M describes a circle with centre at A and radius $\frac{\sqrt{2}}{3}$ (05), when B moves along $y = x + 1$ (05) 15

Aliter :

Since B lies on $y = x + 1$ we can write B = (a, a+1), where a is a parameter.

The equation of AB is $y - 4 = \frac{4 - (a+1)}{2 - a}(x - 2), a \neq 2$

i.e. $y - 4 = \left(\frac{3-a}{2-a}\right)(x - 2)$

Let $M = (x_1\ y_1)$ Then $N = \left(\frac{3x_1 + 7}{6}, \frac{3y_1 + 11}{6}\right)$

$MG \perp AB \Rightarrow \left(\frac{y_1 - \frac{11}{3}}{x_1 - \frac{7}{3}}\right)\left(\frac{3-a}{2-a}\right) = -1$

$\therefore \frac{3-a}{2-a} = \frac{3x_1 - 7}{3y_1 - 11}$ (05)

$\therefore$ the equation of AB is $y - 4 = \frac{(3x_1 - 7)}{(3y_1 - 11)}(x - 2)$ (05)

Since N lies on AB we have

$\frac{3y_1 + 11}{6} - 4 = \frac{3x_1 - 7}{3y_1 - 11}\left(\frac{3x_1 - 7}{6} - 2\right)$

i.e. $(3y_1 - 11)(3y_1 - 13) = (3x_1 - 7)(3x_1 - 5)$

i.e. $9\left(x_1^2 + y_1^2\right) - 36x_1 - 72y_1 + 178 = 0$

As B moves along $y = x + 1$ M describe the circle

$x^2 + y^2 - 4x - 8y + \frac{178}{9} = 0$ which has centre at A radius

$$\sqrt{4+16-\frac{178}{9}}=\frac{\sqrt{2}}{3}$$ (05)

When a = 2, AB is the vertical line $x = 2$

MG is parallel to the x - axis and has equation $y=\frac{11}{7}$

$\therefore$ $G=\left(\frac{7}{3},\frac{11}{3}\right)$, $N=\left(2,\frac{11}{3}\right)$

If $M=(x_1, y_1)$ Then $\frac{x_1+\frac{7}{3}}{2}=2$ and $y=\frac{11}{3}$

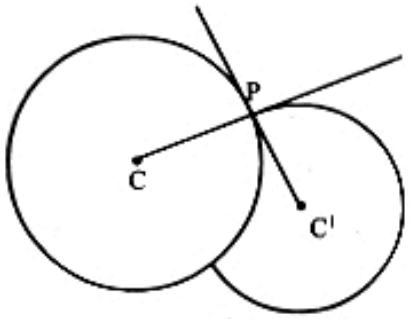
$\therefore$ $M=\left(\frac{5}{3},\frac{11}{3}\right)$, Which satisfies

$$x^2+y^2-4x-8y+\frac{178}{9}=0$$

Hence, in this case also M lies on the circle given by

$$x^2+y^2-4x-8y+\frac{178}{9}=0$$ [15]

(08)



$CC^{12} = CP^2 + CP^{12}$ (05)

$(g - g^1)^2 + (f - f^1)^2 = (g^2 + f^2 - c) + (g^{12} + f^{12} - C^1)$ (10)

$2gg^1 + 2ff^1 = C + C^1$ (05) [20]

$x^2 + y^2 + 4x + 2\lambda y - 6 = (x+2)^2 + y^2 + 2\lambda y - 10 = 0$ (10)

for all values of λ, $(-2 + \sqrt{10}, 0)$ and $(-2 - \sqrt{10}, 0)$ satisfy the equation $(x+2)^2 + y^2 + 2\lambda y - 10 = 0$ (05)

Hence the system of circles represented by $x^2 + y^2 + 4x + 2\lambda y - 6 = 0$ Passes through the points $(-2 + \sqrt{10}, 0)$ and $(-2 - \sqrt{10}, 0)$ (05) [20]

Let $S \equiv x^2 + y^2 + 4x + 2\lambda_0 y - 6 = 0$ be a circle in this system.

Suppose that this circle is or thogonal to a circle given by $x^2 + y^2 + 4x + 2\lambda y - 6 = 0$ (05)

Then $2.2.2 + 2\lambda_0.\lambda = -6 - 6$ (05) i.e. $\lambda = \frac{-10}{\lambda_0}$ (05)

There is only one value for λ and hence there is only one circle which is or thogonal to the circle given by

$S \equiv x^2 + y^2 + 4x + 2\lambda_0 y - 6 = 0$ (05)

$S^1 \equiv x^2 + y^2 + 4x - \frac{10}{\lambda_0} y - 6 = 0$ [20]

$S \equiv x^2 + y^2 + 4x + 4y - 6 = 0 \Rightarrow \lambda_0 = 1$ (05)

$\therefore$ $S^1 \equiv x^2 + y^2 + 4x - 10y - 6 = 0$ (05) [10]

Let $S'' \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ (05)

$S = 0$ and $S^1 = 0$ are or thogonal to $S'' = 0$

If $2.2.g - 2.5.f = c - 6$ and (05)

$2.2.g + 2.2.f = c - 6$ which implies $f = 0$ (05) (05)

Thus the circle $S'' \equiv x^2 + y^2 + 2gx + 4g + 6 = 0$ is orthogonal to both $S = 0$ and $S^1 = 0$

i.e. $S'' \equiv x^2 + y^2 + 2g(x+2) + 6 = 0$ is the general equatoin (05)

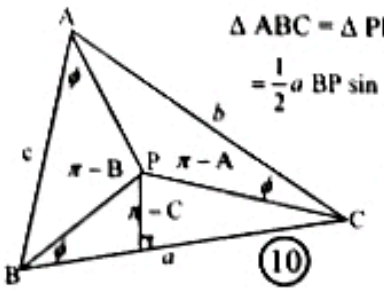
of the circle otthogonal to both $S = 0$ and $S^1 = 0$ (05) [30]

(09) (a)

Sine rule

For any triangle ABC $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$ or

$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ (05) [05]



$\Delta ABC = \Delta PBC + \Delta PCA + \Delta PAB$ (05)

$= \frac{1}{2} a\, BP \sin\phi + \frac{1}{2} b\, CP \sin\phi + \frac{1}{2} c\, AP \sin\phi$ (05)

$= \frac{abc}{2}\left[\frac{BP}{bc}+\frac{CP}{ac}+\frac{AP}{ab}\right]\sin\phi$ (05) [25]

(10)

applying sin rule to the triangle APB we have

$$\frac{BP}{\sin\phi} = \frac{C}{\sin(\pi - B)} \quad (05)$$

$$\therefore BP = \frac{C}{\sin B} \sin\phi \quad (05)$$

applying sin rule to the other two triangles we can show that

$CP = \frac{a}{\sin C} \sin\phi$ *and* $AP = \frac{b}{\sin A} \sin\phi$ (05)

$$\because \text{ABC } \Delta = \frac{abc}{2}\left(\frac{BP}{bc} + \frac{CP}{ac} + \frac{AP}{ab}\right)\sin\phi$$

$$= \left(\frac{1}{2}\frac{ac}{\sin B} + \frac{1}{2}\frac{ab}{\sin C} + \frac{1}{2}\frac{bc}{\sin A}\right) \quad (05)$$

$$= \Delta \text{ ABC} \left[\frac{1}{\sin^2 B} + \frac{1}{\sin^2 C} + \frac{1}{\sin^2 A}\right] \sin^2\phi \ (05)$$

$$\frac{1}{\sin^2\phi} = \frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C} \ (05) \qquad \boxed{30}$$

(b) (i) Let $\tan^{-1}\left(\frac{1}{5}\right) = \alpha$

Then $\tan\alpha = \frac{1}{5}$ (05)

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{5}{12} \ (05)$$

$$2\alpha = \tan^{-1}\left(\frac{5}{12}\right)$$

i.e. $2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{5}{12}\right)$ (05) $\qquad \boxed{15}$

(ii) $\tan 4\alpha = \frac{2\tan 2\alpha}{1 - \tan^2 2\alpha} = \frac{\frac{5}{6}}{1 - \frac{25}{144}} = \frac{120}{119}$ (05)

$$4\alpha = \tan^{-1}\left(\frac{120}{119}\right)$$

i.e. $2\tan^{-1}\left(\frac{5}{12}\right) = \tan^{-1}\left(\frac{120}{119}\right)$ (05) $\qquad \boxed{10}$

(iii) $\tan\left(4\alpha - \frac{\pi}{4}\right) = \frac{\tan 4\alpha - \tan\frac{\pi}{4}}{1 + \tan 4\alpha \cdot \tan\frac{\pi}{4}}$

$$= \frac{\frac{120}{119} - 1}{1 + \left(\frac{120}{119}\right)} = \frac{1}{239} \ (05)$$

$$4\alpha - \frac{\pi}{4} = \tan^{-1}\left(\frac{1}{239}\right)$$

i.e. $\tan^{-1}\left(\frac{120}{119}\right) - \frac{\pi}{4} = \tan^{-1}\left(\frac{1}{239}\right)$ (05) $\qquad \boxed{10}$

$$4\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4} = \tan^{-1}\left(\frac{1}{239}\right)$$

i.e. $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \frac{\pi}{4}$ (05) $\qquad \boxed{05}$