G.C.E. (Advanced Level) Examination - April 2006
Combined Mathematics - I
Provisional Scheme of Marking

(01) (a) $px^2 + qx + r = p\left[x^2 + \frac{q}{p}x + \frac{x}{p}\right]$ (05)

$= p\left\{\left(x + \frac{q}{2p}\right)^2 + \frac{r}{p} - \frac{q^2}{4p^2}\right\}$ (05)

$= p\left\{\left(x + \frac{q}{2p}\right)^2 - \frac{\left(q^2 - 4pr\right)}{4p^2}\right\}$

Thus the equation $px^2 + qx + r = 0$ can be expressed as

$$p\left\{\left(x + \frac{q}{2p}\right)^2 - \frac{\left(q^2 - 4pr\right)}{4p^2}\right\} = 0$$

i.e. $\left(x + \frac{q}{2p}\right)^2 = \left\{\frac{q^2 - 4pr}{4p^2}\right\}$ (05)

Hence the condition for the equation $px^2 + qx + r = 0$ to have coincident roots is $q^2 - 4pr = 0$ (05) [20]

Since the quadratic equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ has coincident roots, $b^2(c-a)^2 - 4ac(b-c)(a-b) = 0$, (10)

i.e. $b^2(c^2 - 2ac + a^2) - 4ac(ab - b^2 - ac + bc) = 0$ (05)

i.e. $b^2c^2 - 2b^2ac + a^2b^2 - 4a^2bc + 4a^2c^2 + 4b^2ac - 4abc^2 = 0$

i.e. $b^2c^2 + 2b^2ac + a^2b^2 - 4a^2bc + 4a^2c^2 + 4b^2ac - 4abc^2 = 0$ (05)

i.e. $b^2c^2 + a^2b^2 + 4a^2c^2 + 2b^2ac - 4a^2bc - 4abc^2 = 0$

i.e. $(bc + ab - 2ac)^2 = 0$ (10)

i.e. $\left[\frac{1}{a} + \frac{1}{c} - \frac{2}{b}\right]^2 = 0$ (05)

Hence, $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ (05) [40]

(b) Let $E = a^3(b - c) + b^3(c - a) + c^3(a - b)$

For fixed b and C, E is a function of a, let us denote if by $f(a)$

Then $f(b) = 0$ and $f(c) = 0$ (05)

Also, (05)

$f(-b-c) = -(b + c)^3(b - c) + b^3(c + b + c) + c^3(-b - c - b)$ (05)

$= (b + c)\{(b + c)^2(c - b) + b^3 - c^3\} +$

$b^3c - bc^3$

$= (b + c)(b - c)\{-(b+c)^2 + (b^2 + bc + c^2) + bc\}$

$= (b + c)(b - c)\{-b^2 - 2bc - c^2 - b^2 + bc + c^2 + bc\}$

$= 0$ (05)

Therefore, by the factor Theorem $(a - b)$, $(a - c)$ and $(a + b + c)$ are factors of $f(a)$. (05)

Therefore, $f(a) = \lambda(a - b)(a - c)(a + b + c)$, where λ is to be determined. (05)

Equation the coefficients of a^3 on either side, we have $\lambda = (b - c)$ (05)

$\therefore$ $E = (a - b)(b - c)(a - c)(a + b + c)$ and the factors of E are $(a - b)$, $(a - c)$, $(b - c)$ and $(a + b + c)$ (05) [40]

(02) (a) (i) (05)

The number of ways of selecting 7 out of 12 children is ${}^{12}C_7$

For each such selection of 7 children into one group the remaining children from the other group of 5 (05)

$\therefore$ The required number of ways ${}^{12}C_7 = \frac{12!}{5!\ 7!}$

$= \frac{12 \times 11 \times 10 \cdot 9 \times 8}{5 \times 4 \times 3 \times 2}$ (05)

$= 792$ [15]

(ii) The number of ways of selecting 6 out of 12 children is ${}^{12}C_6$ (05)

When a group of 6 children is selected the remaining children also from a group of 6, which is also included in ${}^{12}C_6$ Hence, the possible number of selections would be only (05)

half of the above number. (05)

$\therefore$ The required number of ways $= \frac{1}{2}\,{}^{12}C_6$

$= \frac{1}{2} \times \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2}$

$= 462$ (05) [20]

(iii) First take out the tallest child and the shortest child from the 12 children. (05)

Next, select 4 from the remaining 10 children. By adding the tallest and shortest children to each of these groups of 4 children, we can form a group of 6 children containing the tallest child and the shortest child. (05)

$\therefore$ The required number of ways $= {}^{10}C_4$

$= \frac{10!}{4!\ 6!}$

$= \frac{10\times9\times8\times7}{4\times3\times2}$

$= 210$ (05) [15]

(b) $(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$

Where ${}^nC_r = \frac{n!}{r!(n-r)!}$, for $r = 0,1,2,\ldots,n$ (05) (05) [10]

$3^{2n+1} = 3(3^2)^n = 3(9)^n = 3(x+y)^n$, Where $x = 2, y = 7$

Now, (05) (05) (05)

$3(2+7)^n = 3\left(2^n\right) + 3\sum_{r=0}^{n-1} {}^nC_r\, 7^{n-r}\cdot 2^r = 3\left(2^n\right) + 7(k)$ (05) (05)

Where $k = 3\sum_{r=0}^{n-1} {}^nC_r \cdot 7^{n-r-1}\cdot 2^r \in \mathbb{Z}$ [25]

In view of the above

$3^{2n+1} + 2^{n+2}$ (05) $= 7k + 3(2^n) + 2^{n+2}$ (05)

$= 7k + 3(2^n) + 4(2^n)$

$= 7k + 7(2^n)$

$= 7(k + 2^n)$

Hence $3(2^n) + 2^{n+2}$, is divisible by 7. (05) [15]

(03) (a)

When $n = 1$, $P^{n+1} + (P+1)^{2n-1} = P^2 + P + 1$

Which is divisible $= P^2 + P + 1$ $(\neq 0$ for $P \in \mathbb{Z})$

$\therefore$ The assertion is true for $n = 1$. (05)

Assume that the assertion, is true for $n = k \in \mathbb{Z}^+$

i.e. $P^{k+1} + (P+1)^{2k-1} = r(P^2 + P + 1)$ for some $r \in \mathbb{Z}$ (05)

$P^{k+2} + (P+1)^{2k+1}$ (05)

$= P\cdot P^{k+1} (P+1)^2\ (P+1)^{2k-1}$

$= P\left[P^{k+1} + (P+1)^{2k+1}\right] + \left(P^2 + P + 1\right)(P+1)^{2k-1}$

$= \left(P^2 + P + 1\right)\left[rP + (P+1)^{2k-1}\right]$ (05)

Since, $rP + (P+1)^{2k-1} \in \mathbb{Z}^+$ we have $p^2 + p + 1$ divides $p^{k+2} + (p+1)^{2k+1}$ (05)

i.e. If the assertion is true for k then it is true for k+1 where $k \in \mathbb{Z}^+$ (05)

Hence, by the principle of Mathematical Induction the assertion is true for all positive integrain. (05) [35]

(b) $U_r = \frac{r}{1+r^2+r^4}$ (05) [05]

(i) Now

$\left(1+r^2+r\right)\left(1+r^2-r\right) = \left(1+r^2\right)^2 - r^2 = 1+r^2+r^4$ (05)

$\therefore U_r = \frac{r}{\left(1+r+r^2\right)\left(1-r+r^2\right)} = \frac{1}{2}\left[\frac{1}{1-r+r^2} - \frac{1}{1+r+r^2}\right]$ (05) (05)

$\therefore f(r) = \frac{1}{1-r+r^2}$ (05) [20]

Aliter $U_r = \frac{r}{1+r^2+r^4}$ (05) $= \frac{1}{2}\left[f(r) - \frac{1}{1+r+r^2}\right]$

$\therefore f(r) = \frac{2r}{1+r^2+r^4} + \frac{1}{1+r+r^2}$ (05)

$= \frac{2r}{\left(1+r+r^2\right)\left(1-r+r^2\right)} + \frac{1}{1+r+r^2}$ (05)

$= \frac{1}{1+r+r^2}\left[\frac{2r}{1-r+r^2} + 1\right] = \frac{1}{1-r+r^2}$ (05) [20]

(ii) $f(r+1)=\dfrac{1}{1-(r+1)+(r+1)^2}=\dfrac{1}{1+r+r^2}$ (05)

$U_r=\dfrac{1}{2}\left[f(r)-f(r+1)\right]$ (05) [10]

(iii) r = 1, $U_1=\dfrac{1}{2}\left[f(1)-f(2)\right]$

r = 2, $U_2=\dfrac{1}{2}\left[f(2)-f(3)\right]$ (05)

⋮

⋮

r = n-1 $U_{n-1}=\dfrac{1}{2}\left[f(n-1)-f(n)\right]$

r = n, $U_n=\dfrac{1}{2}\left[f(n)-f(n+1)\right]$ (05)

$\therefore \sum_{r=1}^{n} U_r=\dfrac{1}{2}\left[f(1)-f(n+1)\right]$ (05) (05)

$=\dfrac{1}{2}\left[1-\dfrac{1}{1-(n+1)+(n+1)^2}\right]$ (05)

$=\dfrac{1}{2}\left[1-\dfrac{1}{n^2+n+1}\right]=\dfrac{n(n+1)}{2(n^2+n+1)}$ (05)

[30]

(04) (a)

$\dfrac{\cos\alpha+i\sin\alpha}{\cos\beta+i\sin\beta}$

$=\dfrac{(\cos\alpha+i\sin\alpha)}{\cos\beta+i\sin\beta}\times\dfrac{(\cos\beta-i\sin\beta)}{\cos\beta-i\sin\beta}$ (05)

$=\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta+i(\sin\alpha\times\cos\beta-\cos\alpha\sin\beta)}{(\cos^2\beta+\sin^2\beta)}$ (05)

$=\cos(\alpha-\beta)+i\sin(\alpha-\beta)\rightarrow(01)$ (05)

[15]

$\dfrac{z_1}{z_2}=\dfrac{-1+i}{1+i\sqrt{3}}\times\dfrac{1-i\sqrt{3}}{1-i\sqrt{3}}$ (05)

$=\dfrac{1}{4}(\sqrt{3}-1)+\dfrac{i}{4}(\sqrt{3}+1)$

$\mathrm{Re}\,\dfrac{z_1}{z_2}=\dfrac{1}{4}(\sqrt{3}-1)\rightarrow(02)$ (05)

and $\mathrm{Im}\,\dfrac{z_1}{z_2}=\dfrac{1}{4}(\sqrt{3}+1)$ (05) [15]

$z_1=\sqrt{2}\left[\dfrac{-1}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}\right]=\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$ (05)

$z_2=2\left[\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\right]=2\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$ (05) [10]

Now by (01), we have

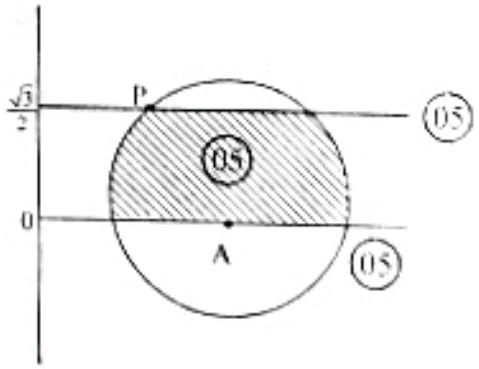
$\dfrac{z_1}{z_2}=\dfrac{\sqrt{2}}{2}\left\{\cos\left(\dfrac{3\pi}{4}-\dfrac{\pi}{3}\right)+i\sin\left(\dfrac{3\pi}{4}-\dfrac{\pi}{3}\right)\right\}$ (05)

$\therefore \mathrm{Re}\,\dfrac{z_1}{z_2}=\dfrac{1}{\sqrt{2}}\cos\dfrac{5\pi}{12}\rightarrow(03)$ (05)

By (02) and (03), We have $\dfrac{1}{\sqrt{2}}\cos\dfrac{5\pi}{12}=\dfrac{1}{4}(\sqrt{3}-1)$

$\therefore \cos\dfrac{5\pi}{12}=\dfrac{1}{4}(\sqrt{6}-\sqrt{2})$ (05) [15]

(b)



(05) (05) (05)

Let P be the point shown in the diagram and let Z_0 be the complex number represented by p.

Also, let A be the center of the circle $|z-2|=1$

Then $\mathrm{Im}\ z_0=\dfrac{\sqrt{3}}{2}\rightarrow(01)$ (05)

and $|z_0 - 2| = 1 \rightarrow (02)$ (05)

Now $(01) \Rightarrow z_0 = x_0 + i\dfrac{\sqrt{3}}{2}$ for some $1 < x_0 < 2$

and $(02) \Rightarrow (x_0 - 2)^2 + \dfrac{3}{4} = 1$ (05)

$\therefore x_0 = \dfrac{3}{2}$ as $1 < x_0 < 2$ and so $z_0 = \dfrac{3}{2} + i\dfrac{\sqrt{3}}{2}$ (05)

Next we note that $OP = |z_0| = \sqrt{\dfrac{9}{4} + \dfrac{3}{4}} = \sqrt{3}$ (05)

Since $OP^2 + AP^2 = 3 + 1$

$= OA^2$, $\hat{OPA} = \dfrac{\pi}{2}$

Thus OP is tangent to the circle at P. (05)

$\therefore$ Arg z becomes greatest over R at P.

$\therefore$ The required complex number $= z_0 = \dfrac{3}{2} + i\dfrac{\sqrt{3}}{2}$ [45]

(05) (a) (i) Since $y = (1 + 4x^2)\tan^{-1} 2x$

$$\frac{dy}{dx} = \underset{(05)}{\left(1 + 4x^2\right)}\frac{2}{1 + 4x^2} + \underset{(05)}{8x\tan^{-1}(2x)}$$

$$= 2 + 8x\tan^{-1}(2x)$$

$$= 2 + \frac{8xy}{\left(1 + 4x^2\right)}$$ (05)

$$\therefore \left(1 + 4x^2\right)\frac{dy}{dx} = 2\left(1 + 4x^2\right) + 8xy$$

$$\left(1 + 4x^2\right)\frac{dy}{dx} - 8xy = 2\left(1 + 4x^2\right) \rightarrow (01)$$ (05) [20]

(ii) $(01) \Rightarrow \left(1 + 4x^2\right)\dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} - 8x\dfrac{dy}{dx} - 8y = 2(8x)$ (05) (05)

$$\Rightarrow \left(1 + 4x^2\right)\frac{d^2y}{dx^2} - 8y = 16x \rightarrow (02)$$ [10]

$$(02) \Rightarrow \left(1 + 4x^2\right)\frac{d^3y}{dx^3} + 8x\frac{d^2y}{dx^2} - 8\frac{dy}{dx} = 16$$ (05)

Substitution of $x = 0$ in this equation gives us

$$\left(\frac{d^3y}{dx^3}\right)_{x=0} - 8\left(\frac{dy}{dx}\right)_{x=0} = 16$$ (05)

If follows from (01) that

$$\left(\frac{dy}{dx}\right)_{x=0} = 2.\left(\frac{d^3y}{dx^3}\right)_{x=0} = 32$$ (05) (05) [20]

(b) Let r cm be the radius and h cm be the height of the cylinder. Then $\pi r^2 h = 1024\pi$ (05)

$\therefore h = \dfrac{1024}{r^2}$ (05)

The total surface area $S = 2\pi rh + 2\pi r^2$ (05)

$$= 2\pi\left[\frac{1024}{r} + r^2\right] r > 0$$ (05)

Then $\dfrac{ds}{dr} = 2\pi\left[\dfrac{-1024}{r^2} + 2r\right]$ (05)

Now $\dfrac{ds}{dr} = 0 \Leftrightarrow \dfrac{1024}{r^2} = 2r \Leftrightarrow r^3 = 512 \Leftrightarrow r = 8$ (05)

Also $\dfrac{ds}{dr} < 0$ When $0 < r < 8$ and (05)

$\dfrac{ds}{dr} > 0$ When $r > 8$ (05)

$\therefore$ The radius that will make the total surface area a minimum is 8cm. (05) [50]

(06) (a) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3 + 2\cos x + \sin x} \rightarrow (01)$

Write $t = \tan\dfrac{x}{2}$ Then $\sin x = \dfrac{2t}{1 + t^2}, \cos x = \dfrac{1 - t^2}{1 + t^2}$ (05)

and $\dfrac{dx}{dt} = \dfrac{2}{\sec^2\dfrac{x}{2}} = \dfrac{2}{1 + t^2}$ (05)

Using these in (01), We have

$$I = \int_0^1 \frac{\left(\dfrac{2}{1 + t^2}\right)dt}{3 + \dfrac{2\left(1 - t^2\right)}{1 + t^2} + \dfrac{2t}{1 + t^2}} = \int_0^1 \frac{2dt}{3\left(1 + t^2\right) + 2\left(1 - t^2\right) + 2t}$$

$$= 2\int_0^1 \frac{dt}{t^2+2t+5} = 2\int_0^1 \frac{dt}{(t+1)^2+4} \quad (10)\ (05)$$

$$= \left[\frac{2}{2}\tan^{-1}\left(\frac{t+1}{2}\right)\right]_0^1 \quad (05)$$

$$= \left(\frac{\pi}{4} - \tan^{-1}\frac{1}{2}\right) \quad (05)$$

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(b) Let $I = \int e^{4x} \sin 3x\, dx$

Integrating by parts we have.

$$I = \frac{1}{4}\int \sin 3x \frac{d}{dx}\left(e^{4x}\right)dx \quad (05)$$

$$= \frac{1}{4}e^{4x}\sin 3x - \frac{1}{4}\int e^{4x}\cdot 3\cos 3x\, dx \quad (05)$$

$$= \frac{1}{4}e^{4x}\sin 3x - \frac{3}{4^2}\int \cos 3x \frac{d}{dx}\left(e^{4x}\right)dx \quad (05)$$

$$= \frac{1}{4}e^{4x}\cdot\sin 3x - \frac{3}{4^2}e^{4x}\cos 3x - \frac{3^2}{4^2}\int e^{4x}\sin 3x\, dx \quad (05)$$

$$= \frac{e^{4x}}{4^2}(4\sin 3x - 3\cos 3x) - \frac{9}{16}I$$

Hence, $\frac{25}{16}I = \frac{e^{4x}}{16}(4\sin 3x - 3\cos 3x)$ (05)

i.e $I = \frac{e^{4x}}{25}(4\sin 3x - 3\cos 3x)$

Hence $\frac{1}{25}(4\sin 3x - 3\cos 3x)$ is an antiderivative of $e^{4x}\sin 3x$.

$\therefore \int e^{4x}\sin 3x\, dx = \frac{e^{4x}}{25}(4\sin 3x - 3\cos 3x) + C$ (05)

Where C is an arbitrary constant. 30

(c) Let $\frac{1}{x^3+1} \equiv \frac{A}{x+1} + \frac{Bx+c}{x^2-x+1}$ (05)

Then $1 \equiv A(x^2 - x + 1) + (x + 1)(Bx + c)$

i.e. $\equiv (A+B)x^2 + (B + C - A)x + (A + C)$

Equationg the coefficients of x^2, x and x^0

We have

$A + B = 0$, $B + C - A = 0$ and $A + C = 1$

Solving these three equations. We have

$A = \frac{1}{3}$, $B = \frac{-1}{3}$ and $C = \frac{2}{3}$ (10)

Thus we get

$$\frac{1}{x^3+1} = \frac{1}{3}\left[\frac{1}{(x+1)} - \frac{(x-2)}{x^2-x+1}\right]$$

$$= \frac{1}{3}\left[\frac{1}{x+1} - \frac{\frac{1}{2}(2x-1) - \frac{3}{2}}{x^2-x+1}\right]$$

$$= \frac{1}{3}\left[\frac{1}{x+1} - \frac{1}{2}\cdot\frac{(2x-1)}{x^2-x+1} + \frac{3}{2}\cdot\frac{1}{\left(\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}\right)}\right] \quad (05)$$

Integrating both sides we have

$$\int\frac{dx}{x^3+1} = \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{6}\int\frac{(2x-1)dx}{x^2-x+1} + \frac{1}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{3}\ell n|x+1| - \frac{1}{6}\ell n|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + C$$

$$= \frac{1}{3}\ell n|x+1| - \frac{1}{6}\ell n|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\frac{(2x-1)}{\sqrt{3}} + C$$

(05) (05) (05)

Where C is an arbitrary constant.

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(07) Let the image of the point $P_1(x_1, y_1)$ in the line $Px + qy + r = 0$ be $P(x, y)$. Then P_1P is perpendicular to the line $Px + qy + r = 0$ and the mid-point of P_1P lies on this line. (05)

The equation of the line through P_1, perpendicular to the given line is ,

$$y - y_1 = \frac{q}{p}(x - x_1) \quad (05)$$

This, can be written as

$$\frac{x - x_1}{p} = \frac{y - y_1}{q} = \lambda\,(say) \quad (05)$$

From this we get the coordinates (x, y) of any point on the line P_1P as

$x = x_1 + p\lambda, \quad y = y_1 + q\lambda$ (05)

We determine the value of λ appropriate to p by making use of the fact that the midpoint of P_1P lies on the given line.

Thus $p\left[x_1 + \frac{p\lambda}{2}\right] + q\left[y_1 + \frac{q\lambda}{2}\right] + r = 0$ (05)

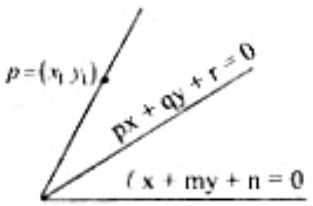
i.e. $\Rightarrow px_1 + qy_1 + r + \frac{\lambda}{2}(p^2 + q^2) = 0$ and hence

$\Rightarrow \lambda = \frac{-2(px_1 + qy_1 + r)}{p^2 + q^2}$ (05)

i.e. The coordinates of P can be written $x = x_1 - p\lambda$, $y = y_1 - q\lambda$, Where

$\lambda = \frac{-2(px_1 + qy_1 + r)}{p^2 + q^2}$ 30

Let P(x,y) be any point on the image of the line $\ell x + my + n = 0$ in the line $px + qy + r = 0$ (05)



Then the image of p in the line $px + qy + r = 0$ lies on the line $\ell x + my + n = 0$ and so (05)

i.e. $\ell(x_1 - p\lambda) + m(y_1 - q\lambda) + n = 0$ (05)

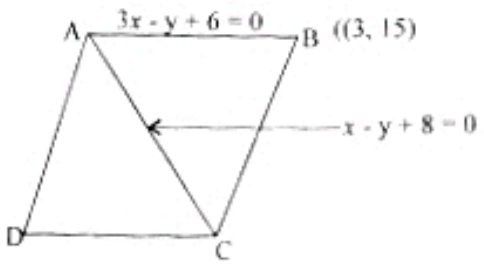
i.e. $\ell x_1 + my_1 + n - \lambda(p\ell + qm) = 0$ (05)

i.e. $(\ell x_1 + my_1 + n) - \frac{-2(p\ell + qm)(px_1 + qy_1 + r)}{p^2 + q^2} = 0$ (05)

Thus the image of the line $\ell x + my + n = 0$ in the line $px + qy + r = 0$ is the line

$\ell x_1 + my_1 + n - \frac{2(p\ell + mq)(px_1 + qy_1 + r)}{p^2 + q^2} = 0$ (05)

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In a rhombus the sides are equal in length and the diagonals cut at right angles. (05)

Therefore, AD is the mirror image of AB on AC using the first part the equation of AD is (05)

$3x - y + 6 - \frac{2(3+1)(x - y + 8)}{2} = 0$ (05)

i.e. $3x - y + 6 - 4(x - y + 8) = 0$

i.e. $x - 3y + 26 = 0$ (05)

Since BC is Paralle to AD and passes through B,

The equation of BC is

$y - 15 = \frac{1}{3}(x - 3)$ (05)

$x - 3y + 42 = 0$

Using thr first part again, the equation of CD is

$x - 3y + 42 - \frac{2(1+3)(x - y + 8)}{2} = 0$ (05)

i.e $x - 3y + 42 - 4(x - y + 8) = 0$ (05)

i.e. $3x - y - 10 = 0$ (05) 40

(08) Let the tangents from $(x_0\ y_0)$ touch the circle $x^2 + y^2 = a^2$ at $(x_1\ y_1)$ and $(x_2\ y_2)$ (05)

The equation of the tangents at $(x_1\ y_1)$ and $(x_2\ y_2)$ are $xx_1 + yy_1 = a^2$ and $xx_2 + yy_2 = a^2$ respectively. (05) (05)

Since, they pass through $(x_0\ y_0)$ we have.

$x_0x_1 + y_0y_1 = a^2$ and $x_0x_2 + y_0y_2 = a^2$ (05)

Thus, the line $x_0x + y_0y = a^2$ passes through $(x_1\ y_1)$ and $(x_2\ y_2)$ (05)

$\therefore$ $x_0x + y_0y = a^2$ is the equation of the chord of contact of the tangents from $(x_0\ y_0)$ (05)

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Let $x^2 + y^2 + 2gx + 2fy + c = 0 \rightarrow (01)$

be the circle through (1, 1) and (-1, 0) (05)

Then $1 + 1 + 2g + 2f + c = 0$ and $1 - 2g + c = 0$ (05)

(05)

i.e. $c = 2g - 1$ and $2 + 2g + 2f + 2g - 1 = 0$ implies $2f = -(4g + 1)$

Then (01) reads as (05)

$x^2 + y^2 + 2gx - (4g + 1)y + 2g - 1 = 0 \rightarrow (02)$ (05)

The equation of the common chord to $S \equiv 0$ and (02) is

$2gx - (4g + 1)y + 2g - 1 + a^2 = 0 \rightarrow (03)$ (05)

Let $R \equiv (x_0\ y_0)$ The equation of the chord of contact of the tangent from R to $S \equiv 0$ is $x_0 x + y_0 y = a^2$, (05)

Which is the same as

$$\therefore \frac{x_0}{2g} = \frac{y_0}{-(4g+1)} = \frac{-a^2}{2g-1+a^2}$$ (05)

(05)

$$\therefore x_0 = \frac{-2ga^2}{2g-1+a^2},\ y_0 = \frac{a^2(4g+1)}{2g-1+a^2}$$ (05)

$2g(x_0 + a^2) = x_0(1 - a^2)$ and $2g(y_0 - 2a^2) = a^2 + y_0(1 - a^2)$ (05)

$$\therefore \frac{x_0(1-a^2)}{x_0+a^2} = \frac{a^2 + y_0(1-a^2)}{y_0 - 2a^2}$$ (05)

i.e. $-2a^2 x_0(1 - a^2) = x_0 a^2 + a^4 - a^2 y_0 (1 - a^2)$

i.e $x_0(-2 + 2a^2 - 1) - y_0(1 - a^2) - a^2 = 0$ $(a \neq 0)$ (05)

$\therefore$ R lies on the line $(2a^2 - 3)x + (a^2 - 1)y - a^2 = 0$ (05)

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(ii) Let $\tan^{-1}\frac{1}{3} = \alpha$ and $\tan^{-1}\frac{1}{7} = \beta$

Then $\tan\alpha = \frac{1}{3}$, $\tan\beta = \frac{1}{7}$, $0 < \alpha < \frac{\pi}{4}$ and $0 < \beta < \frac{\pi}{4}$

Note that $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{3}{1-\frac{1}{9}} = \frac{3}{4}$ (05)

Then $\tan(2\alpha + \beta) = \frac{\tan 2\alpha + \tan\beta}{1 - \tan 2\alpha \tan\beta} = \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4}\cdot\frac{1}{7}} = 1$ (05)

Since $0 < 2\alpha + \beta < \frac{3\pi}{4}$, We get $2\alpha + \beta = \frac{\pi}{4}$ (05)

$$\therefore 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{9}\right) = \frac{\pi}{4} \rightarrow (01)$$ 15

Let $\tan^{-1}\left(\frac{2}{11}\right) = \gamma$ Then $\tan\gamma = \frac{2}{11}$ and $0 < \gamma < \frac{\pi}{4}$

$$\tan(\beta + \gamma) = \frac{\tan\beta + \tan\gamma}{1 - \tan\beta \times \tan\gamma} = \frac{\frac{1}{7} + \frac{2}{11}}{1 - \frac{1}{7}\times\frac{2}{11}} = \frac{25}{75} = \frac{1}{3}$$ (05)

Since $0 < \beta + \gamma < \frac{\pi}{2}$, We get $\beta + \gamma = \tan^{-1}\left(\frac{1}{3}\right)$ (05)

i.e. $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{2}{11}\right) = \tan^{-1}\left(\frac{1}{3}\right) \rightarrow (02)$ 10

(09) (a) (i) $\sin 3\theta = \cos 2\theta \Leftrightarrow \cos\left[\frac{\pi}{2} - 3\theta\right] = \cos 2\theta$ (05)

Thus, $\frac{\pi}{2} - 3\theta = 2n\pi \pm 2\theta$ *where* $n \in \mathbb{Z}$

i.e. $\theta = (1 - 4n)\frac{\pi}{10}$ or $\theta = (1 - 4n)\frac{\pi}{2}$ *for* $n \in \mathbb{Z}$ (05)

Taking n = 0 in the first of these expressions, we see that

$\theta = \frac{\pi}{10} = 18^0$ is a solution of $\sin 3\theta = \cos 2\theta$

Since, $\sin 3\theta = \cos 2\theta$ is equivalent to

$3\sin\theta - 4\sin^3\theta = 1 - 2\sin^2\theta$, We have (05)

$4\sin^3 18^0 - 2\sin^2 18^0 - 3\sin 18^0 + 1 = 0$

$(\sin 18^0 - 1)(4\sin^2 18^0 + 2\sin 18^0 - 1) = 0$ (05)

Since $0 < \sin 18^0 < 1$, this gives us

$$\sin 18^0 = \frac{-2 + \sqrt{20}}{8} = \frac{\sqrt{5} - 1}{4}$$ (05) 25

(01) and (02) give us

$$\frac{\pi}{4} = 2\left[\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{2}{11}\right] + \tan^{-1}\frac{1}{7}$$

$$= 3\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{2}{11}$$ (05) 05

(b) The sine Rule for the triangle ABC is

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, Where the angles A, B, C, and the sides a, b, c are taken in the usual notation.

(05) 05

Let $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$

Then $k^2 = \frac{\sin^2 B + \sin^2 C - \sin^2 A}{b^2 + c^2 - a^2} \rightarrow (01)$ (05)

Now

$$\sin^2 B + \sin^2 C - \sin^2 A = \sin^2 B + \frac{1}{2}(-\cos 2C + \cos 2A)$$

$$= \sin^2 B + \sin(A + C) - \sin(C - A)$$

$$(a s A + C = \pi - B)$$

$$= \sin B[\sin B + \sin(C - A)]$$

$$= \sin B[2 \sin C \ \cos A]$$

$$= 2(bk)(ck)\cos A \quad (10)$$

and (01) *reads as* $1 = \frac{2bc\cos A}{b^2 + c^2 - a^2}$ (05)

i.e $a^2 = b^2 + c^2 - 2bc\cos A$. 20

(i) $\frac{b+c}{5} = \frac{c+a}{6} = \frac{a+b}{7} \Rightarrow \frac{b+c}{5} = \frac{c+a}{6} = \frac{a+b}{7} = \frac{a+b+c}{9}$

$$\Rightarrow \frac{a}{4} = \frac{b}{3} = \frac{c}{2} \quad (05)$$

Since, sin A = ka, sin B = kb and sin C = kc We have

$$\frac{\sin A}{4} = \frac{\sin B}{3} = \frac{\sin C}{2} \quad (05)$$

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(ii) $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 4 - 16}{2 \times 6} = -\frac{1}{4}$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{16 + 4 - 9}{2 \times 8} = \frac{11}{16}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 9 - 4}{2 \times 12} = \frac{7}{8} \quad (10)$$

Hence, $\frac{\cos A}{-1} = \frac{4\cos B}{11} = \frac{2\cos C}{7}$ 10