G.C.E. (Advanced Level) Examination - April 2005
Combined Mathematics - I
Provisional Scheme of Marking

(01) ***(a)*** Since α, β are the roots of $g(x) = 0$.

We have

$\alpha + \beta = -q \rightarrow (01)$ and $\alpha\beta = r \rightarrow (02)$ (05)

$f(\alpha)\, f(\beta) = (\alpha^2 + b\alpha + c)(\beta^2 + b\beta + c)$ (05)

$= \alpha^2\beta^2 + b\alpha^2\beta + c\alpha^2 + b\alpha\beta^2 + b^2\alpha\beta + bc\alpha + c\beta^2 + bc\beta + c^2$

$= \alpha^2\beta^2 + b\alpha\beta(\alpha+\beta) + c[(\alpha+\beta)^2 - 2\alpha\beta] + bc(\alpha+\beta) + b^2\alpha\beta + c^2$ (05)

$= r^2 - brq + c[q^2 - 2r] - bcq + b^2r - c^2$ (05)

$= (c - r)^2 - b(cq - br) + q(cq - br)$

$= (c - r)^2 - (b - q)(cq - br)$ [20]

If $f(x) = 0$ and $g(x) = 0$ have a common root, then

$f(\alpha) = 0$ or $f(\beta) = 0$ and hence $f(\alpha)\, f(\beta) = 0$ (05)

$\therefore$ $(c - r)^2 = (b - q)(cq - br)$ and hence (05)

$b - q$, $c - r$ and $cq - br$ in GP (05) [15]

Since α, γ are root of $f(x) = 0$ we have

$\alpha + \gamma = -b \rightarrow (03)$ and $\alpha\gamma = c \rightarrow (04)$ (05)

By (02) and (04)

$\alpha(\beta + \gamma) = r + c \rightarrow (05)$

and $\alpha^2\beta\gamma = cr \rightarrow (06)$ (05)

Since α is a root of $f(x) = 0$ and $g(x) = 0$ We have

$\alpha^2 + b\alpha + c = 0$ and $\alpha^2 + q\alpha + r = 0$ and hence

$(q - b)\alpha = c - r$

Now $b \neq q$ as $c \neq r$ and hence $\alpha = \dfrac{c-r}{q-b}$ (05)

By (05) $\beta + \gamma = \dfrac{(c+r)(q-b)}{c-r}$ and (05)

by (06) $\beta\gamma = \dfrac{cr(q-r)^2}{(c-r)^2}$ (05)

$\therefore$ The quadratic equation whose roots are β and γ is

$$x^2 - \frac{(c+r)(q-b)x}{(c-r)} + \frac{cr(q-b)^2}{(c-r)^2} = 0$$ [25]

Aliter :

Let α be the common root of $f(x) = 0$ and $g(x) = 0$

Then,

$\alpha^2 + b\alpha + c = 0 \rightarrow (A)$ and $\alpha^2 + q\alpha + r = 0$

Hence $(q - b)\alpha = c - r$, now $b \neq q$ as $c \neq r$ and

Hence $\alpha = \dfrac{c-r}{q-b}$

Now (A) reads as $\left(\dfrac{c-r}{q-b}\right)^2 + b\left(\dfrac{c-r}{q-b}\right) + c = 0$

i.e. $(c - r)^2 = -b(q-b)(c - r) - c(q - b)^2$

$= (b - q)(cq - br)$ and hence $b - q$, $c - r$ and $cq - br$ are in GP.

(b) Let $P(x) = (x - \alpha)\, q(x) + R$, Where $q(x)$ is the quotient and R is the remainder when $P(x)$ is divided by $q(x)$

Then $R = P(\alpha)$ and thus

$P(-1) = -a - b + c = 4 \rightarrow (01)$ (05)

$P(1) = a + b + c = 0 \rightarrow (02)$ (05)

$P(2) = 8a + 2b + c = 4 \rightarrow (03)$ (05)

By (01) and (02), $c = 2$ and $a + b = -2$ (05)

Putting $c = 2$ in (03) we get $4a + b = 1$ (05)

Hence $a = 1$ and $b = -3$ (05) [30]

$p(x) = x^3$ ... $(x-1)(x^2+x-2)$ (05)

$(x-1)^2(x+2)$ (05)

Thus the linear factors of $P(x)$ are $x = 1$ and $x = -2$ [10]

(02) (a)

(i) There are 12 persons and 5 are to be selected to form the team.

This can be done in ${}^{12}C_5$ different ways. (05)

i.e. in $\frac{12\times11\times10\times9\times8}{5\times4\times3\times2} = 11\times72 = 792$ different ways (05) (05) [15]

(ii) Different types of teams can be made to contain 5 girls, 4 girls, 3 girls, 2 girls or 1 girl.

These teams will contain no boys, 1 boy, 2 boys and 4 boys respectively. (05)

The total number of possible teams is

$= 1 + {}^5C_4 \times {}^7C_1 + {}^5C_3 \times {}^7C_2 + {}^5C_2 + {}^7C_3 + {}^5C_1 \times {}^7C_4$

$= 1 + 35 + 10 \times 21 + 10 \times 35$

$= 771$ (10) [15]

Aliter

The number of teams that have no girls is ${}^7C_5 = 21$ (10)

The number of teams with at least one girl is

$792 - 21 = 771$ (05) [15]

(iii) In this case 5 girls can not be selected to form a team but all the other possibilities of (ii) contain at least one girl and at least one boy (10)

∴ the total number of possible teames which contain at least one girl and at least one boy is $771 - 1 = 770$ (10)

[20]

(02) (b) $(1 + 2x + kx^2)^5$ (05)

$= [1 + x(2 + kx)]^5$

$= 1 + 5x(2 + kx) + {}^5C_2x^2(2+kx)^2 + {}^5C_3x^3(2+kx)^3 + {}^5C_4x^4(2+kx)^4 + x^5(2+kx)^5$ (10) (05)

If the coefficient of x^3 is a_3 then $a_3 = {}^5C_2 \times 4k + {}^5C_3 \times 8$

$= 40(k + 2)$ (05) [25]

If $a_3 = 0$ then $k = -2$ (05) [05]

Now, $(1 + 2x - 2x^2)^5 = a_0 + a_1x + \ldots\ldots + a_{10}x^{10}$ (given)

When, $x = 1$, $1 = a_0 + a_1 + \ldots\ldots + a_{10}$ →(01) (05)

When, $x = -1$, $-243 = a_0 - a_1 + a_2 + \ldots\ldots + a_{10}$ →(02) (05)

$\frac{(01)+(02)}{2} \Rightarrow a_0 + a_2 + \ldots\ldots + a_{10} = -121$ (05)

$\frac{(01)-(02)}{2} \Rightarrow a_1 + a_3 + \ldots\ldots + a_9 = 122$ (05) [20]

(03) (a) $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \ldots\ldots\ldots + \frac{1}{n(n+1)(n+2)}$

$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$ →(01)

When n = 1, L.H.S. of (1) = $\frac{1}{6}$ and R.H.S. of (1)

$= \frac{1}{4} - \frac{1}{4} + \frac{1}{6} = \frac{1}{6}$ (05)

There fore, the result is true for n = 1

Let p be any positive integer and assume that the result is true for n = p.

i.e. $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \ldots\ldots\ldots + \frac{1}{p(p+1)(p+2)}$

$= \frac{1}{4} - \frac{1}{2(p+1)} + \frac{1}{2(p+2)}$ (05)

Now consider the case n = p + 1

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots\cdots\frac{1}{p(p+1)(p+2)}+\frac{1}{(p+1)(p+2)(p+3)}$$

$$=\frac{1}{4}-\frac{1}{2(p+1)}+\frac{1}{2(p+2)}+\frac{1}{(p+1)(p+2)(p+3)}\ (05)$$

$$=\frac{1}{4}-\frac{1}{2(p+1)}+\frac{1}{2(p+2)}\cdot\frac{1}{p+2}+\frac{1}{2(p+1)}+\frac{1}{2(p+3)}$$

$$=\frac{1}{4}-\frac{1}{2(p+2)}+\frac{1}{2(p+3)}$$

$$=\frac{1}{4}-\frac{1}{2[(p+1)+1]}+\frac{1}{2[(p+1)+2]}\ (05)$$

Thus the result is true for n = p + 1, if the result is true for n = p

Since the result is true for n = 1, by mathematical Induction the result is true for all positive integral n. (05) [25]

$$\frac{1}{4}-\sum\frac{1}{r(r+1)(r+2)}<\frac{1}{100}\ (05)$$

$$\Leftrightarrow\frac{1}{4}-\left[\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}\right]<\frac{1}{100}$$

$$\Leftrightarrow\frac{1}{n+1}-\frac{1}{n+2)}<\frac{1}{50}$$

$$\Leftrightarrow\frac{(n+2-n-1)}{(n+1)(n+2)}<\frac{1}{50}\Leftrightarrow 50<(n+1)(n+2)\ (05)$$

(05)

Now (n + 1) (n + 2) is an increasing function of n and

When n = 5 we get (5 + 1) (5 +2) < 50 and when n = 6

We get (6 +1) (6 + 2) > 50 (05)

Therefore the smallest integer satisfying the given ineguality is 6. (05) [25]

(b)

$$\frac{1}{2}|x-1|=\begin{cases}\frac{1}{2}(x-1), & \text{if } x\geq 1\\ \frac{1}{2}(x-1), & \text{if } x<1\end{cases}\ (05)$$

$$\text{and } |x-4|=\begin{cases}x-4, & \text{if } x\geq 4\\ -(x-4), & \text{if } x<4\end{cases}\ (05)$$

Case (i) : $x < 1$

$$-\frac{1}{2}(x-1)>-(x-4)\Leftrightarrow x>7\ (05)$$

This is impossible case (ii) $1\leq x<4$ (05)

$$\frac{1}{2}(x-1)>-(x-4)\Leftrightarrow x>3\ (05)$$

Hence in this case we get $3 < x < 4$ (05)

case (iii) : $x \geq 4$

$$\frac{1}{2}(x-1)>x-4\Leftrightarrow x<7\ (05)$$

Hence in this case we get $4\leq x<7$ (05)

Thus the required set of values is $\{x\in\mathbb{R}/3<x<7\}$ (10)

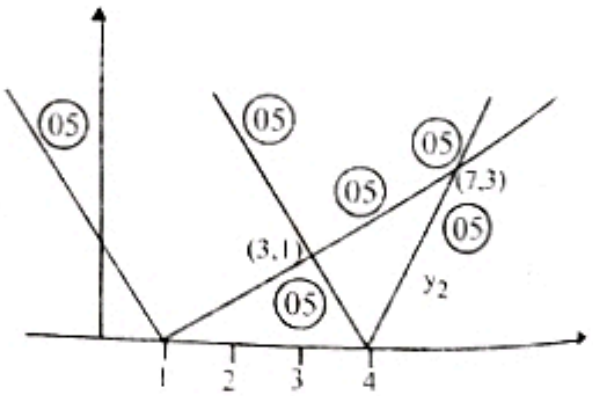
[50]

Aliter :

Let $y_1(x)=\frac{1}{2}|x-1|$ *and* $y_2(x)=|x-4|$

$$y_1(x)=\begin{cases}\frac{1}{2}(x-1), & \text{if } x\geq 1\\ -\frac{1}{2}(x-1), & \text{if } x<1\end{cases}\ (05)$$

$$y_2(x)=\begin{cases}x-4, & \text{if } x\geq 4\\ -(x-4), & \text{if } x<4\end{cases}\ (05)$$

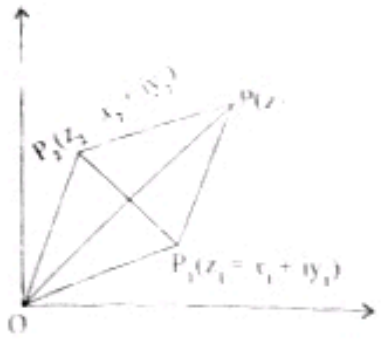


The required set of values $=\{x\in\mathbb{R}/y_1(x)>y_2(x)\}$ (05)

$=\{x\in\mathbb{R}/3<x<7\}$ (05)

[50]

(04)



Let $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ be the point representing the complex number z_1 and z_2 respectively. Let O be the origin. (05)

Bisect P_1P_2 at Q and extend the line OQ to P such that OQ : OP = 1 : 2 (05)

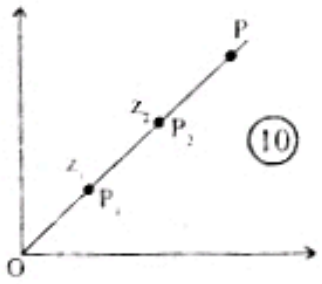
Then the point P represents the complex number $z_1 + z_2$

Since Q is the mid-point of P_1P_2, we have

$$Q \equiv \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right] \quad (05)$$

Since OQ : OP = 1 : 2, We have $P \equiv (x_1 + x_2, y_1 + y_2)$ (05)

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10

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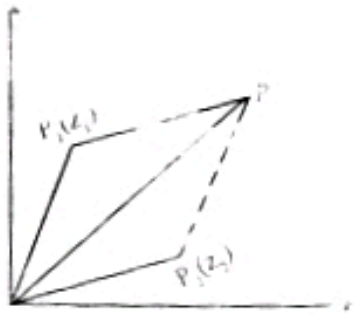
In this case $OP = OP_1 + P_1P$ and $P_1P = OP_2$ Hence

$OP = OP_1 + OP_2$

Same is true when $z_1 = 0$ or $z_2 = 0$

Note that $OP_1 = |z_1|$, $OP_2 = |z_2|$ and that

$OP = |z_1 + z_2|$ (05)



From the triangle OP_1P, we have $OP < OP_1 + P_1P$ (05)

Since OP_1PP_2 is a Parallelogram, $P_1P = OP_2$

Thus $OP < OP_1 + OP_2$ (05)

Same is true when O, P_1 and P_2 are collinear and O lies between P_1 and P_2

15

Since $z_1 = -12 + 5i$, $|z_1| = 13$ (05)

It is given that $|z_2| = 5$.

Now $|z_1 + z_2| \leq |z_1| + |z_2| \Rightarrow |z_1 + z_2| \leq 18$

Equal sing occurs when Arg Z_2 = Arg Z_1

$\therefore$ The greatest value of $|z_1 + z_2|$ is 18 (05)

10

Note that $\frac{\pi}{2} < \text{Arg } z_1 < \pi$ (05)

There fore if $|z_1 + z_2|$ has its greatest value and also

$\frac{\pi}{2} < \text{arg } z_2 < \pi$. Then Arg z = Arg z_1 (05)

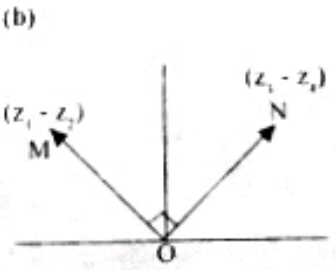
Hence $\frac{|z_2|}{|z_1|} = \frac{OP_2}{OP_1}$ (05)

Since $OP_1 = 13$ and $OP_2 = 5$ we have

$z_2 = \frac{5}{13} z_1 = \frac{-60}{13} + \frac{25}{13} i$ (05)

20

(b)





Let AB be perpendicular to CD, Let M and N be the points representing $z_1 - z_2$ and $z_3 - z_4$ respectively then OM $\perp$ ON (05)

Hence $\left|\text{Arg}(z_1 - z_2) - \text{Arg}(z_3 - z_4)\right| = \frac{\pi}{2}$ (10)

Since $\text{Arg}\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \text{Arg}(z_1 - z_2) - \text{Arg}(z_3 - z_4)$ (05)

We have $\left|\text{Arg}\left(\frac{z_1 - z_2}{z_3 - z_4}\right)\right| = \frac{\pi}{2}$

$\therefore$ $\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$ is purely imaginary. (05) 25

(05) $y = \frac{1}{2}\left(\sin^{-1} x\right)^2$ Then

$\frac{dy}{dx} = 2 \cdot \frac{1}{2} \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}$ *and hence* (05)

$\sqrt{1-x^2}\ \frac{dy}{dx} = \sin^{-1}(x) \rightarrow (01)$ (05)

Differentiating both sides of (01) with respect to x we get,

$\sqrt{1-x^2}\ \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2} \frac{(-2x)}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$ (10)

i.e. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 1 = 0 \rightarrow (02)$ (05) 25

Differentiating both sides of (02) with respect to x we get,

$(1-x^2)\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}(-2x) - \left[x\frac{d^2y}{dx^2} - \frac{dy}{dx}\right] = 0$ (10)

$(1-x^2)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \rightarrow (03)$

Differentiating both sides of (03) with respect to x we get,

$(1-x^2)\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3}(-2x) - 3\left[x\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}\right] - \frac{d^2y}{dx^2} = 0$ (10)

$(1-x^2)\ \frac{d^4y}{dx^4} - 5x\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} = 0 \rightarrow (04)$

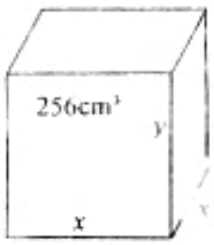
When $x = 0$, from (01), (02), (03) and (04) we get $\frac{dy}{dx} = 0$

$\frac{d^2y}{dx^2} = 1$, $\frac{d^3y}{dx^3} = 0$ *and* $\frac{d^4y}{dx^4} = 4$ respectively

Hence $\left(\frac{dy}{dx}\right)_{x=0} = 0$, $\left(\frac{d^2y}{dx^2}\right)_{x=0} = 1$, $\left(\frac{d^3y}{dx^3}\right)_{x=0} = 0$, (05) (05)

$\left(\frac{d^4y}{dx^4}\right)_{x=0} = 4$ 35

(06)



Area of the base $= x^2$

Cost for the base $= ax^2$

Area of the 4 sides = xy x 4

cost for the 4 sides = 8x . 4xy

Total cost (C) $= 32axy + ax^2$ (05)

$= 32ax \cdot \frac{256}{x^2} + ax^2$

$= 32a \cdot \frac{256}{x} + ax^2 \left(\begin{matrix} since\ x^2y = 256 \\ \Rightarrow y = \frac{256}{x^2} \end{matrix}\right)$ (05)

$\frac{dc}{dx} = \frac{-32 \times 256a}{x^2} + 2ax = 2a\left[x - \frac{16^3}{x^2}\right] = \frac{2a}{x^2}\left(x^3 - 16^3\right)$ (05) (05)

Now $\frac{dc}{dx} = 0 \Rightarrow \frac{2a}{x^2}\left(x^3 - 16^3\right) = 0 \Rightarrow x = 16$ (05) (05)

	$x < 16$	$x = 16$	$x > 16$
$\frac{dc}{dx}$	< 0	$= 0$ (05)	> 0

$\therefore$ C is maximum when $x = 16$ and then

$y = \frac{256}{16^2} = 1$ (05) 40

(06) **(a)** If $\tan\frac{x}{2} = t$ then $\sin x = \frac{2t}{1+t^2}$ and (05)

$$\sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt \Rightarrow dx = \frac{2dt}{1+t^2}$$ (05)

When $x = \frac{\pi}{2}$ we get $t = 1$ and when $x = 0$ we get $t = 0$ (05)

$$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin x} = I = \int_0^1 \frac{2dt}{\left(5+\frac{8t}{1+t^2}\right)\left(1+t^2\right)} = \int_0^1 \frac{2dt}{5t^2+8t+5}$$ (05)

$$= \int_0^1 \frac{2dt}{5t^2+8t+5}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{\left(t+\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}$$ (05)

$$= \frac{2}{5}\cdot\frac{5}{3}\left[\tan^{-1}\frac{\left(t+\frac{4}{5}\right)}{\frac{3}{5}}\right]_0^1$$ (05)

$$= \frac{2}{3}\left[\tan^{-1}3 - \tan^{-1}\frac{4}{3}\right]$$ (05) 35

(b) $\int_0^1 15x^3\sqrt{1+x^2}\,dx$

$$= 15\int_0^1 \frac{x^2}{3}d\left(1+x^2\right)^{\frac{3}{2}}$$ (05)

$$= 5\left(x^2\left(1+x^2\right)^{\frac{3}{2}}\right)_0^1 - 5\int_0^1\left(1+x^2\right)^{\frac{3}{2}}\cdot 2x\cdot dx$$ (05)

(05)

$$= 5\left(2^{\frac{3}{2}}\right) - \frac{5\times 2}{5}\int_0^1 d\left[\left(1+x^2\right)^{\frac{5}{2}}\right]$$ (05)

$$= 5\left(2^{\frac{3}{2}}\right) - 2\left(2^{\frac{5}{2}}-1\right)$$ (05)

(05)

$$= 5\left(2^{\frac{3}{2}}\right) - 4\left(2^{\frac{3}{2}}\right) + 2$$ 30

$$= 2\left(\sqrt{2}+1\right)$$

(c) $\frac{x^2-10x+13}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$ (05)

Then $x^2 - 10x + 13 \equiv A(x-2)^2 + B(x-2)(x-3) + C(x-3)$

When $x = 2$, We have $-3 = -c \Rightarrow c = 3$ (05)

$x = 3$, we have $-8 = A$ (05)

$x = 0$ we have $13 = 4A + 6B - 3c$

$\Rightarrow 6B = 13 - 4A + 3C \Rightarrow 6B = 13 + 32 + 9$

$B = 9$ (05)

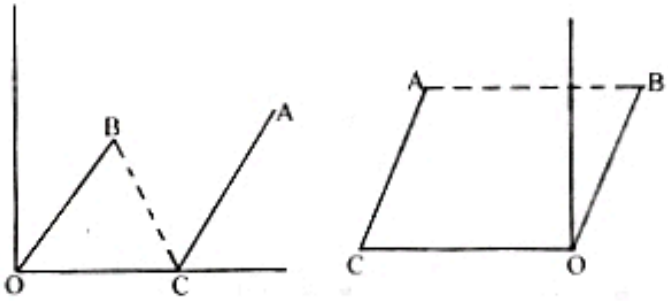
$$\int\frac{x^2-10x+13}{(x-2)\left(x^2-5x+6\right)}dx$$

$$= \int\frac{-8dx}{x-3} + \int\frac{9dx}{x-2} + \int\frac{3dx}{(x-2)^2}$$ (05)

$$= -8ln|x-3| + 9ln|x-2| - \frac{3}{(x-2)^2} + C$$ (05) 35

(05)

(07)



Let $B = \left(a, \frac{4}{3}a\right)$ (05) (as B lies on $4x - 3y = 0$

(05)

and $C \equiv (b,0)$ (as c, lies on $y = 0$)

As $\left(\frac{2}{3}, \frac{2}{3}\right)$ is on BC and the slope of BC is m

We have $\frac{\frac{4}{3}a - \frac{2}{3}}{a - \frac{2}{3}} = m$ (05)

i.e $a\left(\frac{4}{3} - m\right) = \frac{2}{3}(1-m)$ and hence $a = \frac{2(1-m)}{4-3m}$ (10)

Also, $\left(\frac{\frac{2}{3}-0}{\frac{2}{3}-b}\right) = m$ (05) gives $\frac{2}{3} - b = \frac{2}{3m}$ and hence

$b = -\frac{2}{3m}(m-1)$ (05)

$\therefore B \equiv \left(a, \frac{4}{3}a\right)$ and $C \equiv \left[\frac{2}{2m}(\quad\quad)\right]$

Where $a = \frac{2(1-m)}{4(4-3m)}$ 35

$OB^2 = \frac{16}{9}a^2 + a^2 = \frac{25}{9}a^2$ $\quad OB = \left|\frac{5}{3}a\right| = \left|\frac{10(1-m)}{3(4-3m)}\right|$ (10)

$OC = \left|\frac{2(m-1)}{3m}\right|$ (05) 15

If ABOC is a rhombus length OB = lenght OC

$\therefore \frac{10(1-m)}{3(4-3m)} = \pm\frac{2(1-m)}{3m}$ (05) $\therefore \frac{5}{4-3m} = \pm\frac{1}{m}$ (05)

$\therefore m = -2$ or $m = \frac{1}{2}$ (05) 15

The coordinates of the midpoint of BC is $\left(\frac{a+b}{2}, \frac{2a}{3}\right)$ (05)

If $A \equiv (x_0\ y_0)$ the midpoint of OA is $\left(\frac{x_0}{2}, \frac{y_0}{2}\right)$

$\therefore x_0 = a + b = \frac{2}{3m}(m-1) + \frac{2(1-m)}{4-3m} = \frac{4(3m-2)(1-m)}{3m(4-3m)}$ (05)

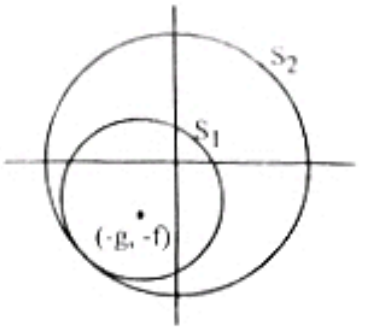
and $y_0 = \frac{4}{3}a = \frac{8(1-m)}{3(4-3m)}$ (05)

When $m = -2$, $x_0 = \frac{8}{5}$ (05) and $y_0 = \frac{4}{5}$ (05) i.e. $A \equiv \left(\frac{8}{5}, \frac{4}{5}\right)$

When $m = \frac{1}{2}$, $x_0 = \frac{-4}{15}$ (05) and $y_0 = \frac{8}{15}$ (05) i.e. $A \equiv \left(\frac{-4}{15}, \frac{8}{15}\right)$

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(08)



Let C_1 be the centre and r_1 be the radius of the circle $S_1 : x^2 + y^2 + 2gx + 2fy = 0$ and let C_2 be the centre and r_2 be the radius of the circle

$S_2 : x^2 + y^2 = 4r^2$ (10)

Then $C_1 = (-g, -f)$, $C_2 = (0, 0)$, $r_1 = \sqrt{g^2 + r^2}$ and $r_2 = 2|r|$

If the circles touch externally then

$C_1C_2 = r_1 + r_2$ (05)

i.e. $\sqrt{g^2 + f^2} = \sqrt{g^2 + f^2} + 2|r|$ and hence $r = 0$ (05)

(a contradiction)

$\therefore$ The circles do not touch externally. 20

The circles touch internally if $C_2C_2 = r_2 - r_1$ (05)

i.e If $\sqrt{g^2 + f^2} = 2r - \sqrt{g^2 + f^2}$ (05)

i.e. $r^2 = g^2 + f^2$ (05) 15

The equation of the common tangent to S_1 and S_2 is $gx + fy + 2r^2 = 0 \rightarrow (01)$ (05)

and the equation of C_1C_2 is $fx - gy = 0 \rightarrow (02)$ (05)

The intersection of these two lines determine the point of contact T of the two circles.

Solving (01) and (02) we get

$x = \frac{-2gr^2}{f^2 + g^2}$ (05) and $y = \frac{-2fr^2}{f^2 + g^2}$ (05)

Thus, $T \equiv \left[\frac{-2gr^2}{f^2 + g^2}, \frac{-2fr^2}{g^2 + f^2}\right] \rightarrow (03)$ 20

Let the equation of the circle which passes through the origin and the point $(a, 0)$ be

$x^2 + y^2 + 2gx + 2fy = 0$ (05)

Then $a^2 + 2ga = 0$, and hence (as $a > 0$) $a = -2g$ (05)

Now $r^2 = 1$ and by the first part the two circle

Will touch internally if $1 = g^2 + f^2$ (05)

This is possible only if $g = \frac{-a}{2}$ and $f = \pm\sqrt{\frac{1-a}{4}}$

i.e. $2f = \pm\sqrt{4-a^2}$ (05)

Thus there are two circles which pass through (a, 0) and the origin and touch $x^2 + y^2 = 4$ (05)

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By (03) the coordinates of the points of contact T1 and T2 are $T_1 = \left[a, \sqrt{4-a^2}\right]$ and $T_2 = \left[a, \sqrt{4-a^2}\right]$ (05)

05

The circle which has T_1 and T_2 as the ends of a diameter is

$$\left(\frac{y-\sqrt{4-a^2}}{x-a}\right)\left(\frac{y+\sqrt{4-a^2}}{x-a}\right) = -1 \quad (10)$$

i.e. $x^2 + y^2 - 2ax + 2a^2 - 4 = 0$ (05)

15

(09) (a) If $\cos 4\theta - \cos 3\theta$

$= 2\cos^2 2\theta - 1 - (4\cos^3\theta - 3\cos\theta)$

(05) (05)

$= 2(2\cos^2\theta - 1)^2 - 1 - 4\cos^3\theta + 3\cos\theta$ (05)

$= 8\cos^4\theta - 4\cos^3\theta - 8\cos^2\theta + 3\cos\theta + 1 \rightarrow (01)$

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If $7\theta = 2n\pi$ for $n \in \mathbb{Z}$ then $4\theta = 2n\pi - 3\theta$ (05)

$\therefore \cos 4\theta = \cos 3\theta$ (05)

10

In this case (01) reads as

$8\cos^4\theta - 4\cos^3\theta - 8\cos^2\theta + 3\cos\theta + 1 = 0$ (05)

This is a 4th degree equation in $\cos\theta = \left[\cos\left(\frac{2n\pi}{7}\right)\right]$,

Whose roots are

$\cos 0,\ \cos\left(\frac{2\pi}{7}\right), \cos\left(\frac{4\pi}{7}\right),$ and $\cos\left(\frac{6\pi}{7}\right)$ (10)

(Taking n = 0,1,2,3)

The sum of the roots is

(ii)

$1 + \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\left(-\frac{4}{8}\right) = \frac{1}{2}$ (05)

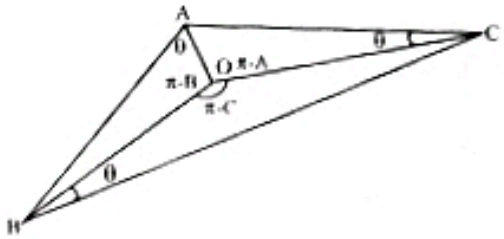
(05)

$\therefore \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$

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(b) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \rightarrow (01)$ (05)

05



Applying the sine rule to Δ OAB

$$\frac{OB}{\sin\theta} = \frac{C}{\sin(\pi - B)} \quad (05)$$

i.e $OB = \frac{C\sin\theta}{\sin B}$ (05)

Applying the same rule to Δ OBC

$$\frac{OB}{\sin(C-\theta)} = \frac{a}{\sin(\pi - C)} \text{ then } OB = \frac{a\sin(C-\theta)}{\sin C} \quad (05)$$

(05)

20

$$\therefore \frac{a(\sin C\sin\theta - \cos C\sin\theta)}{\sin C} = \frac{C\sin\theta}{\sin B}$$

(05)

$$\therefore \cot\theta - \cot C = \frac{C}{a\sin B} = \frac{C}{b\sin A} = \frac{\sin C}{\sin B\cdot\sin A} \quad (05)$$

(05)

$$= \frac{\sin(A+B)}{\sin B\cdot\sin A} = \frac{\sin A\cos B + \cos A\sin B}{\sin B\ \sin A}$$

$= \cot B + \cot A$ (05)

25

$\therefore \cot\theta = \cot A + \cot B + \cot C$