# Combined Mathematics II – 2002 Advanced Level Examination

❖ Answer six questions only.

(01) (a) Brakes are applied to a moving train, at time $t = 0$, giving it a uniform retardation. At $t = 20s$ and at $t = 50s$ its displacements from the position of applying brakes are observed to be $750\ m$ and $1500\ m$ respectively. Sketch a velocity-time graph for the motion of the train, until it comes to rest.

Find

(i) the retardation of the train;

(ii) the velocity of the train at $t = 50\ s$

(iii)the value of $t$, when the train comes to rest.

(b) A motorcyclist is riding due east with constant speed $V$ along a straight level road, and to him a wind blowing with constant velocity appears to blow from the south. When the cyclist doubles his speed, without change of direction, the wind appears to blow from the south-east. Draw velocity triangles for the two situations and find the actual velocity of wind, in magnitude and direction.

(02) (a) A particle is projected under gravity, with initial velocity $u$, making an angle $\alpha$ with the horizontal. After a time $T$, it is at a distance $s$ from the point of projection, moving at right angles to the direction of projection.

Show that

(i) $T = \frac{u}{g \sin \alpha}$

(ii) $S = \frac{1}{2} g T^2$

(b) A train of total mass $300$ metric tons is travelling at a constant speed of $54$ kilometres per hour, on a straight level track, and the total resistance to the motion is $50$ newtons per metric ton. Calculate the power of its engine.

The rear coach, of mass $50$ metric tons, then gets disconnected, but the tractive force of the engine is unaltered.

Find,

(i) the acceleration of the rest of the train;

(ii) the distance moved by the disconnected coach before coming to rest.

[Assume that the motion of this coach is retarded by the resistance alone.]

(03) (a) The masses of their perfectly elastic smooth spheres $A, B, C$ of equal radii are $\lambda m, m, \lambda m$ respectively where $\lambda > 1$. They are placed on a smooth horizontal plane, with their centres in a straight line, in the above order of the letters. Now $B$ is projected towards $A$ with speed $u$, so as to strike $A$ directly.

Show that

(i) the speed of $B$ after this first collision is $\left(\frac{\lambda-1}{\lambda+1}\right) u$ ;

(ii) $B$ will not strike $A$ again if $\lambda \leq 2 + \sqrt{5}$ .

(b) Two masses $3m, m$ are connected by a light inelastic string which passes over a fixed smooth pulley. The system is at rest with the string taut and the portions of the string not in contact with the pulley vertical, the larger mass being on the ground and the smaller mass hanging freely. A third mass $m$, falling vertically from rest, through a height $h$ hits the smaller mass

and adheres to it, jerking the whole system into motion with speed $V$. Find the value of $V$ and show that

(i) the impulse set up in the string is $\frac{3}{5}mu$, where $u = \sqrt{2gh}$

(ii) the larger mass rises to a maximum height $\frac{h}{5}$, after a time $\frac{u}{g}$ from the instant of the jerk.

(04) (a) A particle $P$ is released from rest at a point $A$ on the smooth outer surface of a fixed sphere of centre $O$ and radius $a, OA$ making an acute angle $\alpha$ with the upward vertical. At time $t$, with $P$ still on this surface, $OP$ makes an angle $\theta$ with the upward vertical. Show that

(i) $\dot{\theta}^2 = \frac{2g}{a}(\cos\alpha - \cos\theta)$ ;

(ii) the particle will leave the surface when $\cos\theta = \frac{2}{3}\cos\alpha$ .

(b) A particle $P$ of mass $m$ is tied to the middle point of an elastic string of natural length $2l$ and modulus $mg$. The two ends of the string are attached to two fixed points $A, B$ which are at a distance $4l$ apart on a smooth horizontal table. Initially $A, P, B$ are in a straight line with $AP = 3l$ and $P$ is released from rest, in that position. When the particle $P$ is at a position such that $AP = 2l + x$ write down the equations of motion, and hence obtain the equation $\ddot{x} + \omega^2 x = 0$, where $\omega^2 = \frac{2g}{l}$. Find the centre, amplitude and the period of the simple harmonic motion of $P$.

Find also the maximum speed of the particle and the least time taken to achieve that speed.

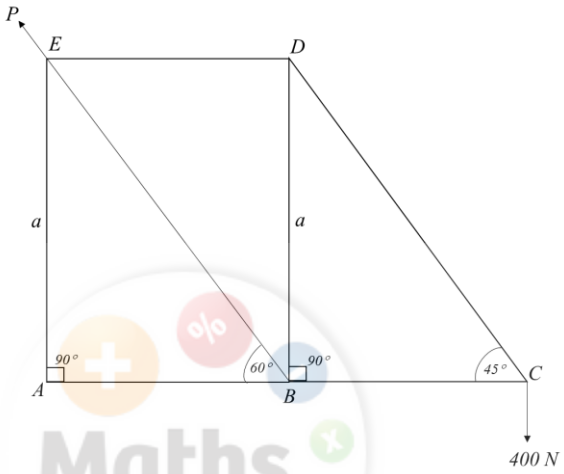
(05) (a) Forces of magnitude $4P, 5P, 6P$ newtons act respectively along the sides $\overline{AB}, \overline{BC}, \overline{CA}$ where $ABC$ is a triangle, right-angled at $A$, with $AB = 4a$ metres and $AC = 3a$ metres. Calculate the magnitude and direction of the resultant of this system of forces, and find the point where its line of action meets $AB$ (produced if necessary).

A couple of moment $M$ in the plane $ABC$, is now added to the system so that the line of action of the resultant of the new system passes through $A$. Find the value of $M$ and the sense of the couple.

(b) Two uniform smooth spheres, each of radius $a$ and weight $W$, lie at rest touching each other inside a fixed smooth hemispherical bowl of radius $b(> 2a)$. Draw, in a separate diagram, a triangle of forces representing forces acting on one of the spheres and show that the reaction between the two spheres is $\frac{Wa}{\sqrt{b(b-2a)}}$ .

(06) (a) Four equal uniform rods weight $W$, are freely jointed at their ends to form a square $ABCD$. The square is hung from the joint $A$, equilibrium with the square shape being maintained by a light rod joining the midpoints of the two lower rods $BC, CD$. Calculate the thrust in the light rod and the reaction at $C$.

(b) The figure below depicts a framework consisting of seven light rods $AB, AE, BC, CD, BD, BE$ and $ED$. The framework is in equilibrium in a vertical plane, with the joint $A$ smoothly pivoted to a fixed point. The joint $C$ carried a load of 400 newtons and a force $P$ newtons, in the direction $\overrightarrow{BE}$ is applied at the joint $E$, so that $ABC$ is horizontal. $AE$ and $BD$ are of equal length $a$ metres ant the angles are as indicated in the figure.




Find the value of $P$ and the horizontal and vertical components of the reaction at the pivot $A$. Hence calculate the stress in each of the rods $AB$ and $AE$ indication whether it is a tension or thrust.

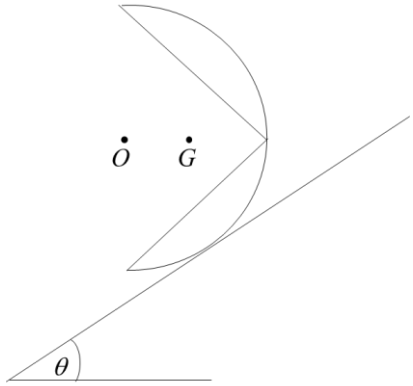
Draw a stress diagram, using Bow's notation for the joint $C$ alone and use it to determine the stress in each of the rods $BC$ and $CD$. Indicate, also, whether it is a tension or a thrust.

(07) Show that the centre of mass of a uniform solid hemisphere of radius $a$ is at a distance $\frac{3}{8}a$ from the base of the hemisphere.

A solid is formed by the removal of a right circular cone of base radius $a$ and height $a$ from a uniform solid hemisphere of radius $a$. The plane bases of the hemisphere and the cone are coincident, with $O$ as the common centre of both. Find the distance from $O$ of the centre of mass $G$ of the solid, assuming that the centre of mass of a right circular cone of height $h$ is at a distance $\frac{3}{4}h$ from the vertex.

The figure shows the vertical cross-section of the above solid resting in equilibrium with a point on the curved surface in contact with a rough plane inclined at an angle $\theta$ to be horizontal. $O$ and

$G$ are in the same vertical plane through a line of greatest slope of the plane. Given that $OG$ is horizontal, show that $\theta = 30°$.




If $W$ is the weight of the hemisphere, obtain in terms of $W$, the values of the fictional force and the normal reaction at the point of contact.

Find also the smallest possible value of the coefficient of friction between the plane and the solid.

(08) If $A$ and $B$ are independent events. Show that the pairs of events $A', B$ and $A', B'$ are also independent, where " $'$ "denotes the complement.

(a) A small airplane is equipped with two engines which operate independently of each other. For a successful flight, at least one of the engines must be working right through the journey. If the probability of and engine failure is $p$ for each of the engines, find in terms of $p$, the probability for a successful flight, taking into consideration engine failures only.

What would be the highest value of $p$ if the probability of a safe flight is to be greater than $0.999999$?

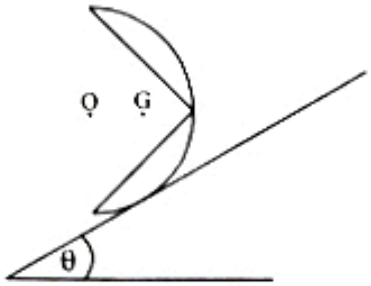
(b) The three characteristics $A, B, C$ can be present in the genes of both male and female adults but only one of those characteristics is found in any one person. The probabilities of a randomly chosen adult to have the characteristics $A, B, C$ are $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ respectively.

(09) Further, the colour of the eyes of a child of a parent with above gene characteristics will be either brown or black and no other colour. The corresponding probabilities for a child's eyes being brown in colour for a given set of parents are as given in the table.

| Father \ mother | A | B | C |
|---|---|---|---|
| A | 0 | 0 | 0 |
| B | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ |
| C | 0 | $\frac{1}{2}$ | 1 |

common centre of both. Find the distance from O of the centre of mass G of the solid, assuming that the centre of mass of a right circular cone of height h is at a distance $\frac{3}{4}h$ from the vertex.

The figure shows the vertical cross-section of the above solid resting in equilibrium with a point on the curved surface in contact with a rough plane inclined at an angle $\theta$ to be horizontal. O and G are in the same vertical plane through a line of greatest slope of the plane. Given that OG is horozontal, show that $\theta = 30^{\circ}$.




If W is the weight of the hemisphere, obtain in terms of W, the values of the fictional force and the normal reaction at the point of contact.

Find also the smallest possible value of the coefficient of friction between the plane and the solid.

(08) If A and B are independent events. show that the pairs of events $A'$, B and $A'$, $B'$ are also independent, where $'$ denotes the complement.

(a) A small airplane is equipped with two engines which operate independently of each other. For a successful flight, at least one of the engines must be working right through the journey. If the **probability of and engine failure** is p for each of the engines, find in terms of p, the probability for a successful flight, taking into consideration engine failures only.

What would be the highest value of p if the probability of a safe flight is to be greater than 0.999999?

(b) The three characteristics A, B, C can be present in the genes of both male and female adults but only one of those characteristics is found in any one person. The probabilities of a randomly chosen adult to have the characteristics A, B, C are $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ respectively.

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Assuming that the presence of A, B, C occur independently among parents,and that, in the usual notation $P(X) = \sum_i P(X|Y_i)\,P(Y_i)$ , find the probability that the colour of the eyes of a randomly chosen child is brown.

It is also known that an aduit possessing characteristic A will only have black eyes and one possessing C will only have brown eyes, while for an adult having characteristic B, the probabilities of the eyes being black or brown in colour are $\frac{3}{4}, \frac{1}{4}$ respectively. Calculate the probability of a child having brown eyes with both parents having black eyes.

(09) In order to estimate the total weight of plantains in a bunch consisting of two hundred plantains, a sample of twenty plantains was randomly selected from a bunch. The weight in grammes of each of the plantains was recorded and the classified readings are given in the following table.

| Class | Class limits (grammes) | Class mark (grammes) | Frequency |
|---|---|---|---|
| 1 | 28-32 | 30 | 7 |
| 2 | 33-37 | 35 | 6 |
| 3 | 38-42 | 40 | 4 |
| 4 | 43-47 | 45 | 2 |
| 5 | 48-52 | 50 | 1 |

By coding the class mark, or otherwise, find the measures: mean, median and mode.

What is the shape of the above distribution?

Also, calculate the variance $S^2$ and hence find the coefficient of skewness.

Suppose that another sample of twenty plantains was taken at random from a bunch of a different variety of plantains containing exactly two hundred plantains.

On weighing and classifying with the same classes as before, the following results were obtained.

| Class | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Frequency | 1 | 2 | 4 | 6 | 7 |

Without making any fresh computations, but justifying your answers, deduce

(i) The shape,
(ii) the variance,
(iii) the mean, median and mode,

of the second distribution.

If a retail trader is interested in buying bunches of plantains from a wholesale trader, what is the most suitable measure that should be used, **agreeable** to both the wholesale and retail traders?