Combined Mathematics I – 2001 Advanced Level Examination

❖ Answer six questions only.

01. (a) Let α and β be the roots of the equation $x^2 + px + 1 = 0$ and let γ and δ be the roots of the equation $x^2 + \frac{1}{P}x + 1 = 0$.

Show that $(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta) = (\gamma^2 + p\gamma + 1)(\delta^2 + p\delta + 1)$ and deduce that $(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta) = \left(P - \frac{1}{P}\right)^2$.

(b) If a and b are positive real number, show that $\log_a b = \frac{1}{\log_b a}$.

show that $\frac{1}{\log_2 2001} + \frac{1}{\log_3 2001} + \frac{1}{\log_4 2001} + \ldots\ldots\ldots \frac{1}{\log_{100} 2001} = \frac{1}{\log_{100!} 2001}$

02. (a) Let $A_{n+1} = (1 - \alpha)(1 - A_n) + A_n$ for $n = 1, 2, 3 \ldots \ldots \ldots$ and $A_1 = \beta$, Where α and β are real numbers. Prove, by the Principle of Mathematical Induction, that for every positive integer n. $A_n = 1 - (1 - \beta)\alpha^{n-1}$.

Find $\sum_{r=1}^{n} A_r$

(b) Show that for integers k and n such that $1 \leq k \leq n$, $K^n C_K = n^{n-1} C_{k-1}$.

Hence or otherwise prove that for any $x \in \mathbb{R}$ and $n \geq 0$,

$\sum_{k=0}^{n} K^n C_k \, x^k (1 - x)^{n-k} = nx$

03. (a) In how many ways can 7 boys and 7 girl be lined up if a girl must be first in line and girls and boys alternate positions in line.

(b) Sketch the graphs of $y = 2|x + 1| - 3$ and $y = x + 2|x - 1|$ in the same diagram.

Hence find the set of x values satisfying $x + 2|x - 1| > 2|x + 1| - 3$.

Solve the equation $x + 2|x - 1| = 2|x + 1| - 3$.

04. (a) If $\arg(z - a) = \alpha$ where $a \in \mathbb{R}$ and $0 < \alpha < \pi$ describe the locus of z.

It is given that $\arg(z + 1) = \frac{\pi}{6}$ and $\arg(z - 1) = \frac{2\pi}{3}$.

Using the first part find z.

(b) Show that the complex number $\frac{5-i}{2-3i}$ can be expressed in the form $\lambda(1 + i)$, where λ is real.

State the value of λ .

Hence, show that $\left(\frac{5-i}{2-3i}\right)^6$ is imaginary and determine its value.

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05. (a) If $x = t - \sin t$ and $y = 1 - \cos t$, show that $y\left(\frac{d^3y}{dx^3}\right) + 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 0$ for $t \neq 2n\pi, n \in \mathbb{Z}$.

(b) Three towns A, B and C are located at the vertices of an isosceles triangle such that the distances from A to B and A to C are equal. The distance from B to C is 12 km and the altitude through A is 16 km.

How far from A, along the altitude through A, should a well be located so that it will use minimum pipe when supplying water to all three towns A,B and C.

06. (a) By making a suitable substitution evaluate the integral $\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$.

(b) By using integration by parts, show that the integral $\int_2^4 x \ln x\,dx = a\ln b + c$ where a, b and c are integers to be determined.

(c) Find $\int_0^1 \frac{(7x-x^2)}{(2-x)(x^2+1)}dx$

07. The straight line $y = mx + c$ intersects the two non-parallel straight lines $u_1 \equiv y - m_1x - c_1 = 0$ and $u_2 \equiv y - m_2x - c_2 = 0$ at A and B respectively. R is a point on AB such that $AR = kRB$. Show that the equation of the straight line joining R to the point of intersection of $u_1 = 0$ and $u_2 = 0$ is $u_1 + \frac{k(m-m_1)}{m-m_2}u_2 = 0$.

The side AB, BC, CA of a triangle ABC lie along the lines $3x + 2y - 6 = 0,\ 2x + y - 2 = 0, x + y - 3 = 0$ respectively. R is a point on AB and Q is a point on AC such that $2\,AR = RB$ and $3AQ = 2QC$.

(i) Find the coordinates of A.

(ii) Write the equations of the lines BQ and CR.

(iii)If BQ and CR meet at D and P is the point of intersection of AD and BC, find the ratio $AP : PB$.

08. Show that the equation of the chord of contact of the tangents drawn from an external point x_0, y_0 to to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_0 + yy_0 + g(x + x_0) + f(y + y_0) + c = 0$.

$x^2 + y^2 + 2x + 6y + 1 = 0$ and $4x + 3y - 5 = 0$ are the equations of a given circle and a given straight line respectively. Show that the line does not circle.

A variable straight line intersects the given circle at two distinct points P and Q, and the tangents to the circle at P and Q meet on the given straight line. Show that this variable line passes through a fixed point, and find the coordinates of this point.

09. (a) Show that for any real number x, $sin^3 2x\ \cos 6x + cos^3 2x\ \sin 6x = \frac{3}{4}\sin 8x$.

Deduce the values of a for which the equation $sin^3 2x\ \cos 6x + cos^3 2x\ \sin 6x = a$ is solvable.

(b) In a triangle, that largest angle is twice size of the smallest angle and the longest side is $1\frac{1}{2}$ times the length of the shortest side. Show that smallest angle of the triangle is $\cos^{-1}\left(\frac{3}{4}\right)$.

Given that length of the middle side is $10\ cm$, find the lengths of the other two sides.